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1 Introduction

A large part of the classical theory of symmetric functions is fairly independent of their interpretation as polynomials in some underlying set of variables X. In fact, if X is supposed infinite, the elementary symmetric functions $\Lambda_k(X)$ are algebraically independent, and what one really considers is just a polynomial algebra $K[\Lambda_1, \Lambda_2, \ldots]$, graded by the weight function $w(\Lambda_k) = k$ instead of the usual degree $d(\Lambda_k) = 1$.

Such a construction still makes sense when the Λ_k are interpreted as noncommuting indeterminates and one can try to lift to the free associative algebra $\mathbf{Sym} = K\langle \Lambda_1, \Lambda_2, \ldots \rangle$ the expressions of the other classical symmetric functions in terms of the elementary ones, in order to define their noncommutative analogs (Section 3.1). Several algebraic and linear bases are obtained in this way, including two families of "power-sums", corresponding to two different noncommutative analogs of the logarithmic derivative of a power series. Moreover, most of the determinantal relations of the classical theory remain valid, provided that determinants be replaced by *quasi-determinants* (*cf.* [GR1], [GR2] or [KL]).

In the commutative theory, Schur functions constitute the fundamental linear basis of the space of symmetric functions. In the noncommutative case, it is possible to define a convenient notion of *quasi-Schur function* (for any skew Young diagram) using quasi-determinants, however most of these functions are not polynomials in the generators Λ_k, but elements of the skew field generated by the Λ_k. The only quasi-Schur functions which remain polynomials in the generators are those which are indexed by *ribbon* shapes (also called *skew hooks*) (Section 3.2).

A convenient property of these noncommutative ribbon Schur functions is that they form a linear basis of $\mathbf{Sym}$. More importantly, perhaps, they also suggest some kind of noncommutative analog of the fundamental relationship between the commutative theory of symmetric functions and the representation theory of the symmetric group. The role of the character ring of the symmetric group is here played by a certain subalgebra Σ_n of its group algebra. This is the *descent algebra*, whose discovery is due to L. Solomon (*cf.* [So]). There is a close connection, which has been known from the beginning, between the product of the descent algebra, and the Kronecker product of representations of the symmetric group. The fact that the homogeneous components of $\mathbf{Sym}$ have the same dimensions as the corresponding descent algebras allows us to transport the product of the descent algebras, thus defining an analog of the usual *internal product* of symmetric functions (Section 5).

Several Hopf algebra structures are classically defined on (commutative or not) polynomial algebras. One possibility is to require that the generators form an infinite sequence of divided powers. For commutative symmetric functions, this is the usual structure, which is precisely compatible with the internal product. The same is true in the noncommutative setting, and the natural Hopf algebra structure of noncommutative symmetric functions provides an efficient tool for computing in the descent algebra. This illustrates once more the importance of Hopf algebras in Combinatorics, as advocated by Rota and his school (see *e.g.* [JR]).

This can be explained by an interesting realization of noncommutative symmetric functions, as a certain subalgebra of the convolution algebra of a free associative algebra (interpreted as a Hopf algebra in an appropriate way). This algebra is discussed at length in the recent book [Re] by C. Reutenauer, where one finds many interesting results which can be immediately translated in the language of noncommutative symmetric functions. We illustrate this correspondence on several examples. In particular, we show that the Lie idempotents in the descent algebra admit a simple interpretation in terms of noncommutative symmetric functions. We also discuss a certain recently discovered family of idempotents of Σ_n, which appear quite naturally when interpreted as noncommutative symmetric functions, and explain to a large extent the combinatorics of Eulerian polynomials.

The algebra of commutative symmetric functions has a canonical scalar product, for which it is self-dual as a Hopf algebra. In the noncommutative theory, the algebra of symmetric functions differs from its dual, which, as shown in [MvR], can be identified with the algebra of quasi-symmetric functions (Section 6).

Another classical subject in the commutative theory is the description of the transition matrices between the various natural bases. This question is considered in Section 4. It is worth noting that the rigidity of the noncommutative theory leads to an explicit description of most of these matrices.

We also investigate the general quasi-Schur functions. As demonstrated in [GR1] or [GR2], the natural object replacing the determinant in noncommutative linear algebra is the quasi-determinant, which is an analog of the ratio of two determinants. Similarly, Schur functions will be replaced by quasi-Schur functions, which are analogs of the ratio of two ordinary Schur functions. The various determinantal expressions of the classical Schur functions can then be adapted to quasi-Schur functions (Section 3.3). This proves useful, for example, when dealing with noncommutative continued fractions and orthogonal polynomials. Indeed, the coefficients of the S-fraction or J-fraction expansions of a noncommutative formal power series are quasi-Schur functions of a special type, as well as the coefficients of the typical three-term recurrence relation for noncommutative orthogonal polynomials.

A rich field of applications of the classical theory is provided by *specializations*. As pointed out by Littlewood, since the elementary functions are algebraically independent, the process of specialization is not restricted to the underlying variables x_i, but can be carried out directly at the level of the Λ_k, which can then be specialized in a totally arbitrary way, and can still be formally considered as symmetric functions of some fictitious set of arguments. The same point of view can be adopted in the noncommutative case, and we discuss several important examples of such specializations (Section 7). The most natural question is whether the formal symmetric functions can actually be interpreted as functions of some set of noncommuting variables. Several answers can be proposed.

In Section 7.1, we take as generating function $\lambda(t) = \sum_k \Lambda_k t^k$ of the elementary symmetric functions a quasi-determinant of the Vandermonde matrix in the noncommutative indeterminates $x_1, x_2, \ldots, x_n$ and $x = t^{-1}$. This is a monic left polynomial of degree n in x, which is annihilated by the substitution $x = x_i$ for every $i = 1, \ldots n$. Therefore the so-defined functions are noncommutative analogs of the ordinary symmetric functions of n commutative variables. They are actually symmetric in the usual sense. These functions are no longer polynomials but rational functions of the x_i. We show that they can be expressed in terms of ratios of quasi-minors of the Vandermonde matrix, as in the classical case. We also indicate in Section 7.2 how to generalize these ideas in the context of skew polynomial algebras.

In Section 7.3, we introduce another natural specialization, namely

$$\lambda(t) = \overleftarrow{\prod_{1\le k\le n}} (1+tx_k) = (1+tx_n)(1+tx_{n-1})(1+tx_{n-2})\cdots(1+tx_1) \ .$$

This leads to noncommutative polynomials which are symmetric for a special action of the symmetric group on the free associative algebra.

In Section 7.4, we take $\lambda(t)$ to be a quasi-determinant of $I+tA$, where $A=(a_{ij})$ is a matrix with noncommutative entries, and I is the unit matrix. In this case, the usual families of symmetric functions are polynomials in the a_{ij} with integer coefficients, and admit a simple combinatorial description in terms of paths in the complete oriented graph labelled by the entries of A. An interesting example, investigated in Section 7.5, is when $A=E_n=(e_{ij})$, the matrix whose entries are the generators of the universal enveloping algebra $U(gl_n)$. We obtain a description of the center of $U(gl_n)$ by means of the symmetric functions associated with the matrices $E_1,\ E_2-I,\ \ldots,\ E_n-(n-1)I$. We also relate these functions to Gelfand-Zetlin bases.

Finally, in Section 7.6, other kinds of specializations in skew polynomial algebras are considered.

The last section deals with some applications of quasi-Schur functions to the study of rational power series with coefficients in a skew field, and to some related points of noncommutative linear algebra. We first discuss noncommutative continued fractions, orthogonal polynomials and Padé approximants. The results hereby found are then applied to rational power series in one variable over a skew field. One obtains in particular a noncommutative extension of the classical rationality criterion in terms of Hankel determinants (Section 8.5).

The n series $\lambda(t)$ associated with the generic matrix of order n (defined in Section 7.4) are examples of rational series. Their denominators appear as n *pseudo-characteristic* polynomials, for which a version of the Cayley-Hamilton theorem can be established (Section 8.6). In particular, the generic matrix posesses n *pseudo-determinants*, which are true noncommutative analogs of the determinant. These pseudo-determinants reduce in the case of $U(gl_n)$ to the Capelli determinant, and in the case of the quantum group $GL_q(n)$, to the quantum determinant (up to a power of q).

The theory of noncommutative rational power series has been initiated by M.P. Schützenberger, in relation with problems in formal languages and automata theory [Sc]. This point of view is briefly discussed in an Appendix.

The authors are grateful to C. Reutenauer for interesting discussions at various stages of the preparation of this paper.

2 Background

2.1 Outline of the commutative theory

Here is a brief review of the classical theory of symmetric functions. A standard reference is Macdonald's book [McD]. The notations used here are those of [LS1].

Denote by $X = \{x_1,\ x_2,\ \dots\}$ an infinite set of *commutative* indeterminates, which will be called a (commutative) *alphabet*. The *elementary symmetric functions* $\Lambda_k(X)$ are then defined by means of the generating series

$$\lambda(X,t) := \sum_{k\geq 0} t^k\, \Lambda_k(X) = \prod_{i\geq 1} (1 + x_i t)\ . \tag{1}$$

The *complete homogeneous symmetric functions* $S_k(X)$ are defined by

$$\sigma(X,t) := \sum_{k\geq 0} t^k\, S_k(X) = \prod_{i\geq 1} (1 - x_i t)^{-1}\ , \tag{2}$$

so that the following fundamental relation holds

$$\sigma(X,t) = \lambda(X,-t)^{-1}\ . \tag{3}$$

The *power sums symmetric functions* $\psi_k(X)$ are defined by

$$\psi(X,t) := \sum_{k\geq 1} t^{k-1}\ \psi_k(X) = \sum_{i\geq 1} x_i\,(1 - x_i t)^{-1}\ . \tag{4}$$

These generating series satisfy the following relations

$$\psi(X,t) = \frac{d}{dt} \log\, \sigma(X,t) = -\frac{d}{dt} \log\, \lambda(X,-t)\ , \tag{5}$$

$$\frac{d}{dt}\, \sigma(X,t) = \sigma(X,t)\ \psi(X,t)\ , \tag{6}$$

$$-\frac{d}{dt}\, \lambda(X,-t) = \psi(X,t)\ \lambda(X,-t)\ . \tag{7}$$

Formula (7) is known as Newton's formula. The so-called fundamental theorem of the theory of symmetric functions states that the $\Lambda_k(X)$ are algebraically independent. Therefore, any formal power series $f(t) = 1 + \sum_{k\geq 1} a_k\, t^k$ may be considered as the specialization of the series $\lambda(X,t)$ to a virtual set of arguments A. The other families of symmetric functions associated to $f(t)$ are then defined by relations (3) and (5). This point of view was introduced by Littlewood and extensively developped in [Li1]. For example, the specialization $S_n = 1/n!$ transforms the generating series $\sigma(X,t)$ into the exponential function e^t. Thus, considering $\sigma(X,t)$ as a symmetric analog of e^t, one can construct symmetric analogs for several related functions, such as trigonometric functions, Eulerian polynomials or Bessel polynomials. This point of view allows to identify their coefficients as the dimensions of certain representations of the symmetric group [F1][F2]. Also, any function admitting a symmetric analog can be given a q-analog, for example by means of the specialization $X = \{\,1, q, q^2, \dots\}$.

We denote by Sym the algebra of symmetric functions, *i.e.* the algebra generated over $\mathbf{Q}$ by the elementary functions. It is a graded algebra for the weight function $w(\Lambda_k) = k$, and the dimension of its homogeneous component of weight n, denoted by Sym_n, is equal to $p(n)$, the number of partitions of n. A *partition* is a finite non-decreasing sequence of positive integers, $I = (i_1 \le i_2 \le \ldots \le i_r)$. We shall also write $I = (1^{\alpha_1} 2^{\alpha_2} \ldots)$, α_m being the number of parts i_k which are equal to m. The *weight* of I is $|I| = \sum_k i_k$ and its *length* is its number of (nonzero) parts $\ell(I) = r$.

For a partition I, we set

$$\psi^I = \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \ , \ \Lambda^I = \Lambda_1^{\alpha_1} \Lambda_2^{\alpha_2} \cdots \ , \ S^I = S_1^{\alpha_1} S_2^{\alpha_2} \cdots \ .$$

For $I \in \mathbf{Z}^r$, not necessarily a partition, the *Schur function* S_I is defined by

$$S_I = \det \left(S_{i_k+k-h} \right)_{1 \le h,k \le r} \tag{8}$$

where $S_j = 0$ for $j < 0$. The Schur functions indexed by partitions form a $\mathbf{Z}$-basis of Sym, and one usually endows Sym with a scalar product $(\cdot,\cdot)$ for which this basis is orthonormal. The ψ^I form then an orthogonal $\mathbf{Q}$-basis of Sym, with $(\psi^I, \psi^I) = 1^{\alpha_1} \alpha_1! \, 2^{\alpha_2} \alpha_2! \cdots$ Thus, for a partition I of weigth n, $n!/(\psi^I, \psi^I)$ is the cardinality of the conjugacy class of S_n whose elements have α_k cycles of length k for all $k \in [1, n]$. A permutation σ in this class will be said *of type* I, and we shall write $T(\sigma) = I$.

These definitions are motivated by the following classical results of Frobenius. Let $CF(\mathbf{S}_n)$ be the ring of central functions of the symmetric group $\mathbf{S}_n$. The *Frobenius characteristic map* $\mathcal{F} : CF(\mathbf{S}_n) \longrightarrow Sym_n$ associates with any central function ξ the symmetric function

$$\mathcal{F}(\xi) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \xi(\sigma) \, \psi^{T(\sigma)} = \sum_{|I|=n} \xi(I) \, \frac{\psi^I}{(\psi^I, \psi^I)}$$

where $\xi(I)$ is the common value of the $\xi(\sigma)$ for all σ such that $T(\sigma) = I$. We can also consider $\mathcal{F}$ as a map from the representation ring $R(\mathbf{S}_n)$ to Sym_n by setting $\mathcal{F}([\rho]) = \mathcal{F}(\chi_\rho)$, where $[\rho]$ denotes the equivalence class of a representation ρ (we use the same letter $\mathcal{F}$ for the two maps since this does not lead to ambiguities). Glueing these maps together, one has a linear map

$$\mathcal{F} : R := \bigoplus_{n \ge 0} R(\mathbf{S}_n) \longrightarrow Sym \ ,$$

which turns out to be an isomorphism of graded rings (see for instance [McD] or [Zel]). We denote by $[I]$ the class of the irreducible representation of S_n associated with the partition I, and by χ_I its character. We have then $\mathcal{F}(\chi_I) = S_I$ (see *e.g.* [McD] p. 62).

The product $*$, defined on the homogeneous component Sym_n by

$$\mathcal{F}([\rho] \otimes [\eta]) = \mathcal{F}(\chi_\rho \chi_\eta) = \mathcal{F}([\rho]) * \mathcal{F}([\eta]) \ , \tag{9}$$

and extended to Sym by defining the product of two homogeneous functions of different weights to be zero, is called the *internal product.*

One can identify the tensor product $Sym \otimes Sym$ with the algebra $Sym(X, Y)$ of polynomials which are separately symmetric in two infinite disjoint sets of indeterminates X

and Y, the correspondence being given by $F\otimes G \longmapsto F(X)\,G(Y)$. Denoting by $X+Y$ the disjoint union of X and Y, one can then define a comultiplication Δ on Sym by setting $\Delta(F) = F(X+Y)$. This comultiplication endows Sym with the structure of a self-dual Hopf algebra, which is very useful for calculations involving characters of symmetric groups (see for instance [Gei], [Zel], [ST] or [Th]). The basic formulas are

$$(F_1F_2\cdots F_r) * G = \mu_r[(F_1\otimes F_2\otimes\cdots\otimes F_r) * \Delta^r G] \ , \tag{10}$$

where μ_r denotes the r-fold ordinary multiplication and Δ^r the iterated coproduct, and

$$\Delta^r(F*G) = \Delta^r(F) * \Delta^r(G) \ . \tag{11}$$

It will be shown in the sequel that both of these formulas admit natural noncommutative analogs. The antipode of the Hopf algebra Sym is given, in λ-ring notation, by

$$\tilde{\omega}(F(X)) = F(-X) \ . \tag{12}$$

The symmetric functions of $(-X)$ are defined by the generating series

$$\lambda(-X,t) := \sigma(X,-t) = [\lambda(X,t)]^{-1} \tag{13}$$

and one can more generally consider *differences* of alphabets. The symmetric functions of $X-Y$ are given by the generating series

$$\lambda(X-Y,t) := \lambda(X,t)\ \lambda(-Y,t) = \lambda(X,t)\ \sigma(Y,-t) \ . \tag{14}$$

In particular, $\psi_k(X-Y) = \psi_k(X) - \psi_k(Y)$.

There is another coproduct δ on Sym, which is obtained by considering products instead of sums, that is

$$\delta(F) = F(XY) \ . \tag{15}$$

One can check that its adjoint is the internal product :

$$(\delta F\ ,\ P\otimes Q) = (F\ ,\ P*Q) \ . \tag{16}$$

This equation can be used to give an intrinsic definition of the internal product, *i.e.* without any reference to characters of the symmetric group. Also, one can see that two bases $(U_I),(V_J)$ of Sym are adjoint to each other iff

$$\sigma(XY,1) = \sum_I U_I(X)\,V_I(Y) \ . \tag{17}$$

For example, writing $\sigma(XY,1) = \prod_i\ \sigma(Y,x_i)$ and expanding the product, one obtains that the adjoint basis of S^I is formed by the *monomial functions* ψ_I.

2.2 Quasi-determinants

Quasi-determinants have been defined in [GR1] and further developed in [GR2] and [KL]. In this section we briefly survey their main properties, the reader being referred to these papers for a more detailed account.

Let K be a field, n an integer and $A = \{ a_{ij},\, 1 \le i,j \le n \}$ an *alphabet* of order n^2, *i.e.* a set of n^2 noncommutative indeterminates. Let $K \not< A \not>$ be the *free field* constructed on K and generated by A. This is the universal field of fractions of the free associative algebra $K\langle A \rangle$ (*cf.* [Co]). The matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called the *generic matrix* of order n. This matrix is invertible over $K \not< A \not>$.

Let A^{pq} denote the matrix obtained from the generic matrix A by deleting the p-th row and the q-th column. Let also $\xi_{pq} = (a_{p1}, \ldots, \hat{a}_{pq}, \ldots, a_{pn})$ and $\eta_{pq} = (a_{1q}, \ldots, \hat{a}_{pq}, \ldots, a_{nq})$.

Definition 2.1 *The quasi-determinant $|A|_{pq}$ of order pq of the generic matrix A is the element of $K \not< A \not>$ defined by*

$$|A|_{pq} = a_{pq} - \xi_{pq}\, (A^{pq})^{-1}\, \eta_{pq} \;\; = a_{pq} - \sum_{i \ne p, j \ne q} a_{pj}\, ((A^{pq})^{-1})_{ji}\, a_{iq} \;.$$

It is sometimes convenient to adopt the following more explicit notation

$$|A|_{pq} = \begin{vmatrix} a_{11} & \ldots & a_{1q} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{p1} & \ldots & \boxed{a_{pq}} & \ldots & a_{pn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \ldots & a_{nq} & \ldots & a_{nn} \end{vmatrix} .$$

Quasi-determinants are here only defined for generic matrices. However, using substitutions, this definition can be applied to matrices with entries in an arbitrary skew field. In fact, one can even work in a noncommutative ring, provided that A^{pq} be an invertible matrix.

Example 2.2 For $n = 2$, there are four quasi-determinants :

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21} \;, \quad \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}\, a_{21}^{-1}\, a_{22} \;,$$

$$\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22}\, a_{12}^{-1}\, a_{11} \;, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21}\, a_{11}^{-1}\, a_{12} \;.$$

The next result can be considered as another definition of quasi-determinants (for generic matrices).

Proposition 2.3 *Let A be the generic matrix of order n and let $B = A^{-1} = (b_{pq})_{1 \le p,q \le n}$ be its inverse. Then one has $|A|_{pq} = b_{qp}^{-1}$ for every $1 \le p,q \le n$.*

It follows from Proposition 2.3 that $|A|_{pq} = (-1)^{p+q} \det A / \det A^{pq}$ when the a_{ij} are commutative variables. Thus quasi-determinants are noncommutative analogs of the ratio of a determinant to one of its principal minors. If A is an invertible matrix with entries in a arbitrary skew field, the above relation still holds for every p,q such that $b_{qp} \ne 0$. Another consequence of Proposition 2.3 is that

$$|A|_{pq} = a_{pq} - \sum_{i \ne p, j \ne q} a_{pj}\, |A^{pq}|_{ij}^{-1}\, a_{iq} \;,$$

which provides a recursive definition of quasi-determinants.

Let I be the unit matrix of order n. The expansions of the quasi-determinants of $I-A$ into formal power series are conveniently described in terms of paths in a graph. Let $\mathcal{A}_n$ denote the complete oriented graph with n vertices $\{1,2,\ldots,n\}$, the arrow from i to j being labelled by a_{ij}. We denote by $\mathcal{P}_{ij}$ the set of words labelling a path in $\mathcal{A}_n$ going from i to j, *i.e.* the set of words of the form $w = a_{ik_1}\, a_{k_1k_2}\, a_{k_2k_3}\, \ldots\, a_{k_{r-1}j}$. A *simple path* is a path such that $k_s \neq i,\ j$ for every s. We denote by $\mathcal{SP}_{ij}$ the set of words labelling simple paths from i to j.

Proposition 2.4 *Let $i,\ j$ be two distinct integers between* 1 *and* n. *Then,*

$$|I-A|_{ii} = 1 - \sum_{\mathcal{SP}_{ii}} w\ , \quad |I-A|_{ij}^{-1} = \sum_{\mathcal{P}_{ji}} w\ . \tag{18}$$

Example 2.5 For $n=2$,

$$\begin{vmatrix} \boxed{1-a_{11}} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} = 1 - a_{11} - \sum_{p\geq 0} a_{12}\, a_{22}^p\, a_{21}\ .$$

As a general rule, quasi-determinants are not polynomials in their entries, except in the following special case, which is particularly important for noncommutative symmetric functions (a graphical interpretation of this formula can be found in the Appendix).

Proposition 2.6 *The following quasi-determinant is a polynomial in its entries :*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & -1 & a_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{n-1\,n} \\ 0 & \ldots & 0 & -1 & a_{nn} \end{vmatrix} = a_{1n} + \sum_{1\leq j_1<j_2<\ldots<j_k<n} a_{1j_1}\, a_{j_1+1\,j_2}\, a_{j_2+1\,j_3}\, \ldots\, a_{j_k+1\,n}\ . \tag{19}$$

We now recall some useful properties of quasi-determinants. Quasi-determinants behave well with respect to permutation of rows and columns.

Proposition 2.7 *A permutation of the rows or columns of a quasi-determinant does not change its value.*

Example 2.8

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & \boxed{a_{31}} & a_{33} \end{vmatrix}\ .$$

One can also give the following analogs of the classical behaviour of a determinant with respect to linear combinations of rows and columns.

Proposition 2.9 *If the matrix B is obtained from the matrix A by multiplying the p-th row* on the left *by λ, then*

$$|B|_{kq} = \begin{cases} \lambda\, |A|_{pq} & \text{for} \quad k=p\ , \\ |A|_{kq} & \text{for} \quad k\neq p\ . \end{cases}$$

Similarly, if the matrix C is obtained from the matrix A by multiplying the q-th column on the right *by μ, then*

$$|C|_{pl} = \begin{cases} |A|_{pq}\,\mu & \text{for} \quad l = q\ , \\ |A|_{pl} & \text{for} \quad l \neq q\ . \end{cases}$$

Finally, if the matrix D is obtained from A by adding to some row (resp. column) of A its k-th row (resp. column), then $|D|_{pq} = |A|_{pq}$ for every $p \neq k$ (resp. $q \neq k$).

The following proposition gives important identities which are called *homological relations*.

Proposition 2.10 *The quasi-minors of the generic matrix A are related by :*

$$|A|_{ij}\,(|A^{il}|_{kj})^{-1} = -|A|_{il}\,(|A^{ij}|_{kl})^{-1}\ ,$$
$$(|A^{kj}|_{il})^{-1}\,|A|_{ij} = -(|A^{ij}|_{kl})^{-1}\,|A|_{kj}\ .$$

Example 2.11

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & \boxed{a_{32}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\ .$$

The classical expansion of a determinant by one of its rows or columns is replaced by the following property.

Proposition 2.12 *For quasi-determinants, there holds :*

$$|A|_{pq} = a_{pq}\ -\ \sum_{j\neq q}\ a_{pj}\,(|A^{pq}|_{kj})^{-1}\,|A^{pj}|_{kq}\ ,$$
$$|A|_{pq} = a_{pq}\ -\ \sum_{i\neq p}\ |A^{iq}|_{pl}\,(|A^{pq}|_{il})^{-1}\,a_{iq}\ ,$$

for every $k \neq p$ and $l \neq q$.

Example 2.13 Let $n = p = q = 4$. Then,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix} = a_{44} - a_{43} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix}$$

$$- a_{42} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix} - a_{41} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & \boxed{a_{34}} \end{vmatrix}\ .$$

Let P, Q be subsets of $\{1,\ldots,n\}$ of the same cardinality. We denote by A^{PQ} the matrix obtained by removing from A the rows whose indices belong to P and the columns whose indices belong to Q. Also, we denote by A_{PQ} the submatrix of A whose row indices belong to P and column indices to Q. Finally, if a_{ij} is an entry of some submatrix A^{PQ} or A_{PQ}, we denote by $|A^{PQ}|_{ij}$ or $|A_{PQ}|_{ij}$ the corresponding quasi-minor.

Here is a noncommutative version of Jacobi's ratio theorem which relates the quasi-minors of a matrix with those of its inverse. This identity will be frequently used in the sequel.

Theorem 2.14 *Let A be the generic matrix of order n, let B be its inverse and let $(\{i\}, L, P)$ and $(\{j\}, M, Q)$ be two partitions of $\{1, 2, \ldots, n\}$ such that $|L| = |M|$ and $|P| = |Q|$. Then, one has*

$$|B_{M\cup\{j\},L\cup\{i\}}|_{ji} = |A_{P\cup\{i\},Q\cup\{j\}}|_{ij}^{-1} \ .$$

Example 2.15 Take $n = 5$, $i = 3$, $j = 4$, $L = \{1, 2\}$, $M = \{1, 3\}$, $P = \{4, 5\}$ and $Q = \{2, 5\}$. There holds

$$\begin{vmatrix} a_{32} & \boxed{a_{34}} & a_{35} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & \boxed{b_{43}} \end{vmatrix}^{-1} .$$

We conclude with noncommutative versions of Sylvester's and Bazin's theorems.

Theorem 2.16 *Let A be the generic matrix of order n and let P, Q be two subsets of $[1, n]$ of cardinality k. For $i \notin P$ and $j \notin Q$, we set $b_{ij} = |A_{P\cup\{i\},Q\cup\{j\}}|_{ij}$ and form the matrix $B = (b_{ij})_{i\notin P, j\notin Q}$ of order $n - k$. Then,*

$$|A|_{lm} = |B|_{lm}$$

for every $l \notin P$ and every $m \notin Q$.

Example 2.17 Let us take $n = 3$, $P = Q = \{3\}$ and $l = m = 1$. Then,

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{Vmatrix} \boxed{\begin{Vmatrix} \boxed{a_{11}} & a_{13} \\ a_{31} & a_{33} \end{Vmatrix}} & \begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} \boxed{a_{21}} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{Vmatrix} .$$

Let A be a matrix of order $n \times p$, where $p \geq n$. Then, for every subset P of cardinality n of $\{1, \ldots, p\}$, we denote by A_P the square submatrix of A whose columns are indexed by P.

Theorem 2.18 *Let A be the generic matrix of order $n \times 2n$ and let m be an integer in $\{1, \ldots, n\}$. For $1 \leq i, j \leq n$, we set $b_{ij} = |A_{\{j,n+1,\ldots,n+i-1,n+i+1,\ldots,2n\}}|_{mj}$ and form the matrix $B = (b_{ij})_{1\leq i,j\leq n}$. Then we have*

$$|B|_{kl} = |A_{\{n+1,\ldots,2n\}}|_{m,n+k} \ |A_{\{1,\ldots,l-1,l+1,\ldots,n,n+k\}}|_{m,n+k}^{-1} \ |A_{\{1,\ldots,n\}}|_{ml}$$

for every integers k, l in $\{1, \ldots, n\}$.

Example 2.19 Let $n = 3$ and $k = l = m = 1$. Let us adopt more appropriate notations, writing for example $|2\boxed{4}5|$ instead of $|M_{\{2,4,5\}}|_{14}$. Bazin's identity reads

$$\begin{vmatrix} \boxed{|\boxed{1}56|} & |\boxed{2}56| & |\boxed{3}56| \\ |\boxed{1}46| & |\boxed{2}46| & |\boxed{3}46| \\ |\boxed{1}45| & |\boxed{2}45| & |\boxed{3}45| \end{vmatrix} = |\boxed{4}56| \ |23\boxed{4}|^{-1} \ |\boxed{1}23| \ .$$

We shall also need the following variant of Bazin's theorem 2.18.

Theorem 2.20 *Let A be the generic matrix of order $n\times(3n-2)$ and let m be an integer in $\{1,\dots,n\}$. For $1\le i,j\le n$, we set $c_{ij}=|A_{\{j,n+i,n+i+1,\dots,2n+i-2\}}|_{mj}$ and form the matrix $C=(c_{ij})_{1\le i,j\le n}$. Then we have*

$$|C|_{11}=|A_{\{n+1,\dots,2n\}}|_{m,2n}\ |A_{\{2,3\dots,n,2n\}}|^{-1}_{m,2n}\ |A_{\{1,\dots,n\}}|_{m1}\quad .$$

Example 2.21 Let $n=3$, $m=1$, and keep the notations of 2.19. Theorem 2.20 reads

$$\begin{vmatrix} \boxed{|\boxed{1}45|} & |\boxed{2}45| & |\boxed{3}45| \\ |\boxed{1}56| & |\boxed{2}56| & |\boxed{3}56| \\ |\boxed{1}67| & |\boxed{2}67| & |\boxed{3}67| \end{vmatrix} = |45\boxed{6}|\ |23\boxed{6}|^{-1}\ |\boxed{1}23|\ . \tag{20}$$

Proof — We use an induction on n. For $n=2$, Theorem 2.20 reduces to Theorem 2.18. For $n=3$, we have to prove (20). To this end, we note that the specialization $7\to4$ gives back Bazin's theorem 2.18. Since the right-hand side does not contain 7, it is therefore enough to show that the left-hand side does not depend on 7. But by Sylvester's theorem 2.16, the left-hand side is equal to

$$\begin{vmatrix} \boxed{|\boxed{1}45|} & |\boxed{2}45| \\ |\boxed{1}56| & |\boxed{2}56| \end{vmatrix} - \begin{vmatrix} |\boxed{2}45| & \boxed{|\boxed{3}45|} \\ |\boxed{2}56| & |\boxed{3}56| \end{vmatrix} \begin{vmatrix} |\boxed{2}56| & |\boxed{3}56| \\ |\boxed{2}67| & \boxed{|\boxed{3}67|} \end{vmatrix}^{-1} \begin{vmatrix} |\boxed{1}56| & |\boxed{2}56| \\ \boxed{|\boxed{1}67|} & |\boxed{2}67| \end{vmatrix}$$

$$= |45\boxed{6}|\ |25\boxed{6}|^{-1}\ |\boxed{1}25| - |45\boxed{6}|\ |25\boxed{6}|^{-1}\ |2\boxed{3}5||2\boxed{3}6|^{-1}|\boxed{1}26|\ ,$$

which is independent of 7. Here, the second expression is derived by means of 2.18 and Muir's law of extensible minors for quasi-determinants [KL]. The general induction step is similar, and we shall not detail it. □

3 Formal noncommutative symmetric functions

In this section, we introduce the algebra **Sym** of formal noncommutative symmetric functions which is just the free associative algebra $K\langle \Lambda_1, \Lambda_2, \ldots \rangle$ generated by an infinite sequence of indeterminates $(\Lambda_k)_{k\geq 1}$ over some fixed commutative field K of characteristic zero. This algebra will be graded by the weight function $w(\Lambda_k) = k$. The homogeneous component of weight n will be denoted by $\mathbf{Sym}_n$. The Λ_k will be informally regarded as the elementary symmetric functions of some virtual set of arguments. When we need several copies of **Sym**, we can give them labels $A, B, \ldots$ and use the different sets of indeterminates $\Lambda_k(A), \Lambda_k(B), \ldots$ together with their generating series $\lambda(A,t), \lambda(B,t), \ldots$, and so on. The corresponding algebras will be denoted $\mathbf{Sym}(A)$, $\mathbf{Sym}(B)$, *etc.*

We recall that a *composition* is a vector $I = (i_1, \ldots, i_k)$ of nonnegative integers, called the *parts* of I. The *length* $l(I)$ of the composition I is the number k of its parts and the weigth of I is the sum $|I| = \sum i_j$ of its parts.

3.1 Elementary, complete and power sums functions

Let t be another indeterminate, commuting with all the Λ_k. It will be convenient in the sequel to set $\Lambda_0 = 1$.

Definition 3.1 *The* elementary *symmetric functions are the Λ_k themselves, and their generating series is denoted by*

$$\lambda(t) := \sum_{k\geq 0} t^k \, \Lambda_k = 1 + \sum_{k\geq 1} t^k \, \Lambda_k \ . \tag{21}$$

The complete homogeneous *symmetric functions S_k are defined by*

$$\sigma(t) := \sum_{k\geq 0} t^k \, S_k = \lambda(-t)^{-1} \ . \tag{22}$$

The power sums *symmetric functions of the* first kind *Ψ_k are defined by*

$$\psi(t) := \sum_{k\geq 1} t^{k-1} \, \Psi_k \ , \tag{23}$$

$$\frac{d}{dt} \, \sigma(t) = \sigma(t) \ \psi(t) \ . \tag{24}$$

The power sums *symmetric functions of the* second kind *Φ_k are defined by*

$$\sigma(t) = \exp\left(\sum_{k\geq 1} t^k \, \frac{\Phi_k}{k} \right) , \tag{25}$$

or equivalently by one of the series

$$\Phi(t) := \sum_{k\geq 1} t^k \, \frac{\Phi_k}{k} = \log\left(1 + \sum_{k\geq 1} S_k \, t^k \right) \tag{26}$$

or

$$\phi(t) := \sum_{k\geq 1} t^{k-1} \, \Phi_k = \frac{d}{dt} \, \Phi(t) = \frac{d}{dt} \, \log \sigma(t) \ . \tag{27}$$

Although the two kinds of power sums coincide in the commutative case, they are quite different at the noncommutative level. For instance,

$$\Phi_3 = \Psi_3 + \frac{1}{4}\left(\Psi_1\,\Psi_2 - \Psi_2\,\Psi_1\right) . \tag{28}$$

The appearance of two families of power sums is due to the fact that one does not have a unique notion of logarithmic derivative for power series with noncommutative coefficients. The two families selected here correspond to the most natural noncommutative analogs. We shall see below that both of them admit interesting interpretations in terms of Lie algebras. Moreover, they may be related to each other via appropriate specializations (*cf.* Note 5.14).

One might also introduce a third family of power sums by replacing (24) by

$$\frac{d}{dt}\,\sigma(t) = \psi(t)\,\sigma(t) \quad ,$$

but this would lead to essentially the same functions. Indeed, **Sym** is equiped with several natural involutions, among which the *anti*-automorphism which leaves invariant the Λ_k. Denote this involution by $F \longrightarrow F^*$. It follows from (22) and (26) that one also has $S_k^* = S_k$, $\Phi_k^* = \Phi_k$, and,

$$\frac{d}{dt}\,\sigma(t) = \psi(t)^*\,\sigma(t) \quad ,$$

with $\psi(t)^* = \sum_{k\geq 1}\, t^{k-1}\,\Psi_k^*$.

Other involutions, to be defined below, send Λ_k on $\pm S_k$. This is another way to handle the left-right symmetry in **Sym**, as shown by the following proposition which expresses $\psi(t)$ in terms of $\lambda(t)$.

Proposition 3.2 *One has*

$$-\,\frac{d}{dt}\,\lambda(-t) = \psi(t)\,\lambda(-t) . \tag{29}$$

Proof — Multiplying from left and right equation (24) by $\lambda(-t)$, we obtain

$$\lambda(-t)\,\left(\frac{d}{dt}\,\sigma(t)\right)\,\lambda(-t) = \sigma(t)^{-1}\,\left(\frac{d}{dt}\,\sigma(t)\right)\,\sigma(t)^{-1} = \psi(t)\,\lambda(-t) .$$

But one also has

$$\sigma(t)^{-1}\,\left(\frac{d}{dt}\,\sigma(t)\right)\,\sigma(t)^{-1} = -\,\frac{d}{dt}\,\sigma(t)^{-1} = -\,\frac{d}{dt}\,\lambda(-t) .$$

□

These identities between formal power series imply the following relations between their coefficients.

Proposition 3.3 *For $n \geq 1$, one has*

$$\sum_{k=0}^{n} (-1)^{n-k} S_k \Lambda_{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \Lambda_k S_{n-k} = 0 , \tag{30}$$

$$\sum_{k=0}^{n-1} S_k \Psi_{n-k} = n S_n , \quad \sum_{k=0}^{n-1} (-1)^{n-k-1} \Psi_{n-k} \Lambda_k = n \Lambda_n . \tag{31}$$

Proof — Relation (30) is obtained by considering the coefficient of t^n in the defining equalities $\sigma(t)\lambda(-t) = \lambda(-t)\sigma(t) = 1$. The other identity is proved similarly from relations (24) and (29). □

In the commutative case, formulas (30) and (31) are respectively called Wronski and Newton formulas.

It follows from relation (26) that for $n \geq 1$

$$\Phi_n = n S_n + \sum_{i_1+\ldots+i_m=n, m>1} c_{i_1,\ldots,i_m} S_{i_1} \ldots S_{i_m} \tag{32}$$

where $c_{i_1,\ldots,i_m}$ are some rational constants, so that Proposition 3.3 implies in particular that **Sym** is freely generated by any of the families (S_k), (Ψ_k) or (Φ_k). This observation leads to the following definitions.

Definition 3.4 *Let $I = (i_1, \ldots, i_n) \in (\mathbf{N}^*)^n$ be a composition. One defines the products of complete symmetric functions*

$$S^I = S_{i_1} S_{i_2} \ldots S_{i_n} . \tag{33}$$

Similarly, one has the products of elementary symmetric functions

$$\Lambda^I = \Lambda_{i_1} \Lambda_{i_2} \ldots \Lambda_{i_n} , \tag{34}$$

the products of power sums of the first kind

$$\Psi^I = \Psi_{i_1} \Psi_{i_2} \ldots \Psi_{i_n} , \tag{35}$$

and the products of power sums of the second kind

$$\Phi^I = \Phi_{i_1} \Phi_{i_2} \ldots \Phi_{i_n} . \tag{36}$$

Note 3.5 As in the classical case, the complete and elementary symmetric functions form **Z**-bases of **Sym** (*i.e.* bases of the algebra generated over **Z** by the Λ_k) while the power sums of first and second kind are just **Q**-bases.

The systems of linear equations given by Proposition 3.3 can be solved by means of quasi-determinants. This leads to the following quasi-determinantal formulas.

Corollary 3.6 *For every $n \geq 1$, one has*

$$S_n = (-1)^{n-1} \begin{vmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_{n-1} & \boxed{\Lambda_n} \\ \Lambda_0 & \Lambda_1 & \dots & \Lambda_{n-2} & \Lambda_{n-1} \\ 0 & \Lambda_0 & \dots & \Lambda_{n-3} & \Lambda_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Lambda_0 & \Lambda_1 \end{vmatrix}, \tag{37}$$

$$\Lambda_n = (-1)^{n-1} \begin{vmatrix} S_1 & S_0 & 0 & \dots & 0 \\ S_2 & S_1 & S_0 & \dots & 0 \\ S_3 & S_2 & S_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boxed{S_n} & S_{n-1} & S_{n-2} & \dots & S_1 \end{vmatrix}, \tag{38}$$

$$n\, S_n = \begin{vmatrix} \Psi_1 & \Psi_2 & \dots & \Psi_{n-1} & \boxed{\Psi_n} \\ -1 & \Psi_1 & \dots & \Psi_{n-2} & \Psi_{n-1} \\ 0 & -2 & \dots & \Psi_{n-3} & \Psi_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -n+1 & \Psi_1 \end{vmatrix}, \tag{39}$$

$$n\, \Lambda_n = \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & 0 \\ \Psi_2 & \Psi_1 & 2 & \dots & 0 \\ \Psi_3 & \Psi_2 & \Psi_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boxed{\Psi_n} & \Psi_{n-1} & \Psi_{n-2} & \dots & \Psi_1 \end{vmatrix}, \tag{40}$$

$$\Psi_n = \begin{vmatrix} S_1 & S_0 & 0 & \dots & 0 \\ 2S_2 & S_1 & S_0 & \dots & 0 \\ 3S_3 & S_2 & S_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boxed{nS_n} & S_{n-1} & S_{n-2} & \dots & S_1 \end{vmatrix} = \begin{vmatrix} \Lambda_1 & 2\Lambda_2 & \dots & (n-1)\Lambda_{n-1} & \boxed{n\Lambda_n} \\ \Lambda_0 & \Lambda_1 & \dots & \Lambda_{n-2} & \Lambda_{n-1} \\ 0 & \Lambda_0 & \dots & \Lambda_{n-3} & \Lambda_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Lambda_0 & \Lambda_1 \end{vmatrix}. \tag{41}$$

Proof — Formulas (37) and (38) follow from relations (30) by means of Theorem 1.8 of [GR1]. The other formulas are proved in the same way. □

Note 3.7 In the commutative theory, many determinantal formulas can be obtained from Newton's relations. Most of them can be lifted to the noncommutative case, as illustrated on the following relations, due to Mangeot in the commutative case (see [LS1]).

$$(-1)^{n-1}\, n\, S_n = \begin{vmatrix} 2\,\Lambda_1 & 1 & 0 & \dots & 0 \\ 4\,\Lambda_2 & 3\,\Lambda_1 & 2\,\Lambda_0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (2n-2)\,\Lambda_{n-1} & (2n-3)\,\Lambda_{n-2} & (2n-4)\,\Lambda_{n-3} & \dots & (n-1)\,\Lambda_0 \\ \boxed{n\,\Lambda_n} & (n-1)\,\Lambda_{n-1} & (n-2)\,\Lambda_{n-2} & \dots & \Lambda_1 \end{vmatrix}. \tag{42}$$

To prove this formula, it suffices to note that, using relations (31), one can show as in the commutative case that the matrix involved in (42) is the product of the matrix

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_1 & 1 & 0 & \dots & 0 \\ -\Psi_2 & \Psi_1 & 2 & \dots & 0 \\ \Psi_3 & -\Psi_2 & \Psi_1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-1}\,\Psi_n & (-1)^{n-2}\,\Psi_{n-1} & (-1)^{n-3}\,\Psi_{n-2} & \dots & \Psi_1 \end{pmatrix}$$

by the matrix $\mathbf{\Lambda} = (\Lambda_{j-i})_{0\le i,j\le n-1}$. Formula (42) then follows from Theorem 1.7 of [GR1].

Proposition 3.3 can also be interpreted in the Hopf algebra formalism. Let us consider the comultiplication Δ defined in **Sym** by

$$\Delta(\Psi_k) = 1 \otimes \Psi_k + \Psi_k \otimes 1 \tag{43}$$

for $k \ge 1$. Then, as in the commutative case (*cf.* [Gei] for instance), the elementary and complete functions form infinite sequences of divided powers.

Proposition 3.8 *For every $k \ge 1$, one has*

$$\Delta(S_k) = \sum_{i=0}^{k} S_i \otimes S_{k-i}\ , \quad \Delta(\Lambda_k) = \sum_{i=0}^{k} \Lambda_i \otimes \Lambda_{k-i}\ .$$

Proof — We prove the claim for $\Delta(S_k)$, the other proof being similar. Note first that there is nothing to prove for $k = 0$ and $k = 1$. Using now induction on k and relation (31), we obtain

$$k\,\Delta(S_k) = \sum_{i=0}^{k-1}\sum_{j=0}^{i} (S_j \otimes S_{i-j}\Psi_{k-i} + S_{i-j}\Psi_{k-i} \otimes S_j)$$

$$= \sum_{j=0}^{k-1} S_j \otimes (\sum_{i=j}^{k-1} S_{i-j}\Psi_{k-i}) \ + \ \sum_{j=0}^{k-1} (\sum_{i=j}^{k-1} S_{i-j}\Psi_{k-i}) \otimes S_j$$

$$= \sum_{j=0}^{k-1} (S_j \otimes (k-j)\,S_{k-j}) \ + \ \sum_{j=0}^{k-1} (k-j)\,S_{k-j} \otimes S_j = \ k \sum_{j=0}^{k} (S_j \otimes S_{k-j})\ .$$

□

We define an anti-automorphism ω of **Sym** by setting

$$\omega(S_k) = \Lambda_k \tag{44}$$

for $k \ge 0$. The different formulas of Corollary 3.6 show that

$$\omega(\Lambda_k) = S_k\ , \quad \omega(\Psi_k) = (-1)^{k-1}\,\Psi_k\ .$$

In particular we see that ω is an involution. To summarize:

Proposition 3.9 *The comultiplication Δ and the antipode $\tilde{\omega}$, where $\tilde{\omega}$ is the anti-automorphism of* **Sym** *defined by*

$$\tilde{\omega}(S_k) = (-1)^k \Lambda_k, \quad k \geq 0 \ , \tag{45}$$

endow **Sym** *with the structure of a Hopf algebra.*

The following property has some interesting consequences.

Proposition 3.10 *The Lie algebra generated by the family (Φ_k) coincides with the Lie algebra generated by the family (Ψ_k). We shall denote it by $L(\Psi)$. Moreover the difference $\Phi_k - \Psi_k$ lies in the Lie ideal $L^2(\Psi)$ for every $k \geq 1$.*

Proof — In order to see that the two Lie algebras $L(\Psi)$ and $L(\Phi)$ coincide, it is sufficient to show that Φ lies in $L(\Psi)$. According to Friedrichs' criterion (*cf.* [Re] for instance), we just have to check that Φ is primitive for the comultiplication Δ. Using (26) and Proposition 3.8, we can now write

$$\sum_{k\geq 1} t^k \, \frac{\Delta(\Phi_k)}{k} = \ \log \big(\sum_{k\geq 0} \ \Delta(S_k) \, t^k \, \big) = \log \big(\, UV \, \big) \ , \tag{46}$$

where we respectively set $U = \sum_{k\geq 0} \, (1 \otimes S_k) \, t^k$ and $V = \sum_{k\geq 0} \, (S_k \otimes 1) \, t^k$. Since all coefficients of U commute with all coefficients of V, we have $\log(UV) = \log(U) + \log(V)$ from which, applying again (26), we obtain

$$\sum_{k\geq 1} \ t^k \, \frac{\Delta(\Phi_k)}{k} = \sum_{k\geq 1} \ t^k \, \frac{1 \otimes \Phi_k}{k} \ + \ \sum_{k\geq 1} \ t^k \, \frac{\Phi_k \otimes 1}{k} \ , \tag{47}$$

as required. The second point follows from the fact that $\Phi_k - \Psi_k$ is of order at least 2 in the Ψ_i, which is itself a consequence of (32). □

Example 3.11 The first Lie relations between Φ_k and Ψ_k are given below.

$$\Phi_1 = \Psi_1, \quad \Phi_2 = \Psi_2, \quad \Phi_3 = \Psi_3 + \frac{1}{4}\,[\Psi_1, \Psi_2], \quad \Phi_4 = \Psi_4 + \frac{1}{3}\,[\Psi_1, \Psi_3],$$

$$\begin{aligned} \Phi_5 = \Psi_5 + \frac{3}{8}\,[\Psi_1, \Psi_4] + \frac{1}{12}\,[\Psi_2, \Psi_3] + \frac{1}{72}\,[\Psi_1, [\Psi_1, \Psi_3]] \\ + \frac{1}{48}\,[[\Psi_1, \Psi_2], \Psi_2] + \frac{1}{144}\,[[[\Psi_2, \Psi_1], \Psi_1], \Psi_1] \ . \end{aligned}$$

It is worth noting that, according to Proposition 3.10, the Hopf structures defined by requiring the Φ_k or the Ψ_k to be primitive are the same. Note also that the antipodal property of $\tilde{\omega}$ and the fact that

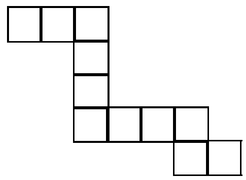
$$\Delta(\Phi_k) = 1 \otimes \Phi_k + \Phi_k \otimes 1 \tag{48}$$

show that we have

$$\omega(\Phi_k) = (-1)^{k-1} \, \Phi_k \ . \tag{49}$$

3.2 Ribbon Schur functions

A general notion of *quasi-Schur function* can be obtained by replacing the classical Jacobi-Trudi determinant by an appropriate quasi-determinant (see Section 3.3). However, as mentioned in the introduction, these expressions are no longer polynomials in the generators of **Sym**, except in one case. This is when all the subdiagonal elements $a_{i+1,i}$ of their defining quasi-determinant are scalars, which corresponds to Schur functions indexed by *ribbon shaped* diagrams. We recall that a *ribbon* or *skew-hook* is a skew Young diagram containing no 2×2 block of boxes. For instance, the skew diagram $\Theta = I/J$ with $I = (3,3,3,6,7)$ and $J = (2,2,2,5)$



is a ribbon. A ribbon Θ with n boxes is naturally encoded by a *composition* $I = (i_1, \dots, i_r)$ of n, whose parts are the lengths of its rows (starting from the top). In the above example, we have $I = (3,1,1,4,2)$. Following [MM], we also define the *conjugate composition* $I^\sim$ of I as the one whose associated ribbon is the conjugate (in the sense of skew diagrams) of the ribbon of I. For example, with $I = (3,1,1,4,2)$, we have $I^\sim = (1,2,1,1,4,1,1)$.

In the commutative case, the skew Schur functions indexed by ribbon diagrams possess interesting properties and are strongly related to the combinatorics of compositions and descents of permutations (*cf.* [Ge], [GeR] or [Re] for instance). They have been defined and investigated by MacMahon (*cf.* [MM]). Although the commutative ribbon functions are not linearly independent, *e.g.* $R_{12} = R_{21}$, it will be shown in Section 4 that the noncommutative ones form a linear basis of **Sym**.

Definition 3.12 *Let $I = (i_1, \dots, i_n) \in (\mathbf{N}^*)^n$ be a composition. Then the* ribbon *Schur function R_I is defined by*

$$R_I = (-1)^{n-1} \begin{vmatrix} S_{i_1} & S_{i_1+i_2} & S_{i_1+i_2+i_3} & \dots & \boxed{S_{i_1+\dots+i_n}} \\ S_0 & S_{i_2} & S_{i_2+i_3} & \dots & S_{i_2+\dots+i_n} \\ 0 & S_0 & S_{i_3} & \dots & S_{i_3+\dots+i_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & S_{i_n} \end{vmatrix} \tag{50}$$

As in the commutative case, we have $S_n = R_n$ and $\Lambda_n = R_{1^n}$ according to Corollary 3.6. Moreover, the ribbon functions $R_{(1^k,n-k)}$ play a particular role. In the commutative case, they are the *hook Schur functions*, denoted $S_{(1^k,n-k)}$. We shall also use this notation for the noncommutative ones.

We have for the multiplication of ribbon Schur functions the following formula, whose commutative version is due to MacMahon (*cf.* [MM]).

Proposition 3.13 *Let $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$ be two compositions. Then,*

$$R_I \, R_J = R_{I \triangleright J} + R_{I \cdot J}$$

where $I \triangleright J$ denotes the composition $(i_1, \dots, i_{r-1}, i_r + j_1, j_2, \dots, j_s)$ and $I \cdot J$ the composition $(i_1, \dots, i_r, j_1, \dots, j_s)$.

Proof — Using the above notations, we can write that R_I is equal to

$$(-1)^{k-1} \begin{vmatrix} S_{i_1} & \dots & \boxed{S_{i_1+\dots+i_{k-1}}} \\ S_0 & \dots & S_{i_2+\dots+i_{k-1}} \\ \vdots & & \vdots \\ 0 & \dots & S_{i_{k-1}} \end{vmatrix} \begin{vmatrix} S_0 & \dots & S_{i_2+\dots+i_{k-1}} \\ 0 & \dots & S_{i_3+\dots+i_{k-1}} \\ \vdots & & \vdots \\ 0 & \dots & \boxed{S_0} \end{vmatrix}^{-1} \begin{vmatrix} S_{i_1} & S_{i_1+i_2} & \dots & S_{i_1+\dots+i_k} \\ S_0 & S_{i_2} & \dots & S_{i_2+\dots+i_k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boxed{S_{i_k}} \end{vmatrix} ,$$

$$= R_{(i_1,\dots,i_{k-1})} \left(S_{i_k} - R^{-1}_{(i_1,\dots,i_{k-1})} R_{(i_1,\dots,i_{k-2},i_{k-1}+i_k)} \right) ,$$

the first equality following from the homological relations between quasi-determinants of the same matrix, and the second one from the expansion of the last quasi-determinant by its last row together with the fact that the second quasi-determinant involved in this relation is equal to 1. Hence,

$$R_I \, R_n = R_{I \triangleright n} + R_{I \cdot n} \ . \tag{51}$$

Thus, the proposition holds for $\ell(J) = 1$. Let now $J = (j_1, \dots, j_n)$ and $J' = (j_1, \dots, j_{n-1})$. Then relation (51) shows that

$$R_J = R_{J'} \, R_{j_n} - R_{J' \triangleright j_n} \ .$$

Using this last identity together with induction on $\ell(J)$, we get

$$R_I \, R_J = R_{I \triangleright J'} \, R_{j_n} + R_{I . J'} \, R_{j_n} - R_{I \triangleright (J' \triangleright j_n)} - R_{I \cdot (J' \triangleright j_n)} \ ,$$

and the conclusion follows by means of relation (51) applied two times. □

Proposition 3.13 shows in particular that the product of an elementary by a complete function is the sum of two hook functions,

$$\Lambda_k \ S_l = R_{1^k l} + R_{1^{k-1}(l+1)} \ . \tag{52}$$

We also note the following expression of the power sums Ψ_n in terms of hook Schur functions.

Corollary 3.14 *For every $n \geq 1$, one has*

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k \, R_{1^k (n-k)} \ .$$

Proof — The identity $\lambda(-t) \, \dfrac{d}{dt} \, \sigma(t) = \psi(t)$ implies that

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k \, (n-k) \, \Lambda_k \, S_{n-k} \ , \tag{53}$$

and the result follows from (52). □

Let us introduce the two infinite Toeplitz matrices

$$\mathbf{S} = (S_{j-i})_{i,j \geq 0} \ , \quad \mathbf{\Lambda} = \left((-1)^{j-i} \, \Lambda_{j-i} \right)_{i,j \geq 0} \ , \tag{54}$$

where we set $S_k = \Lambda_k = 0$ for $k < 0$. The identity $\lambda(-t)\ \sigma(t) = \sigma(t)\ \lambda(-t) = 1$ is equivalent to the fact that $\mathbf{\Lambda}\ \mathbf{S} = \mathbf{S}\ \mathbf{\Lambda} = I$. We know that the quasi-minors of the inverse matrix A^{-1} are connected with those of A by Jacobi's ratio theorem for quasi-determinants (Theorem 2.14). Thus the ribbon functions may also be expressed in terms of the Λ_k, as shown by the next proposition.

Proposition 3.15 *Let $I \in \mathbf{N}^n$ be a composition and let $I^\sim = (j_1, \ldots, j_m)$ be the conjugate composition. Then one has the relation*

$$R_I = (-1)^{m-1} \begin{vmatrix} \Lambda_{j_m} & \Lambda_{j_{m-1}+j_m} & \Lambda_{j_{m-2}+j_{m-1}+j_m} & \cdots & \boxed{\Lambda_{j_1+\ldots+j_m}} \\ \Lambda_0 & \Lambda_{j_{m-1}} & \Lambda_{j_{m-2}+j_{m-1}} & \cdots & \Lambda_{j_1+\ldots+j_{m-1}} \\ 0 & \Lambda_0 & \Lambda_{j_{m-2}} & \cdots & \Lambda_{j_1+\ldots+j_{m-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_{j_1} \end{vmatrix} .$$

Proof — This formula is obtained by applying Jacobi's theorem for quasi-determinants (Theorem 2.14) to the definition of R_I. □

Corollary 3.16 *For any composition I, one has*

$$\omega(R_I) = R_{I^\sim} .$$

3.3 Quasi-Schur functions

We shall now consider general quasi-minors of the matrices $\mathbf{S}$ and $\mathbf{\Lambda}$ and proceed to the definition of quasi-Schur functions. As already mentioned, they are no longer elements of **Sym**, but of the free field $K \not< S_1, S_2, \ldots \not>$ generated by the S_i.

Definition 3.17 *Let $I = (i_1, i_2, \ldots, i_n)$ be a* partition, *i.e. a weakly increasing sequence of nonnegative integers. We define the* quasi-Schur function *$\check{S}_I$ by setting*

$$\check{S}_I = (-1)^{n-1} \begin{vmatrix} S_{i_1} & S_{i_2+1} & \cdots & \boxed{S_{i_n+n-1}} \\ S_{i_1-1} & S_{i_2} & \cdots & S_{i_n+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i_1-n+1} & S_{i_2-n+2} & \cdots & S_{i_n} \end{vmatrix} . \qquad (55)$$

In particular we have $\check{S}_i = S_i$, $\check{S}_{1^i} = \Lambda_i$ and $\check{S}_{1^i(n-i)} = R_{1^i(n-i)}$. However it must be noted that for a general partition I, the quasi-Schur function $\check{S}_I$ reduces in the commutative case to the *ratio* of two Schur functions S_I/S_J, where $J = (i_1 - 1, i_2 - 1, \ldots, i_{n-1} - 1)$. One can also define in the same way *skew Schur functions*. One has to take the same minor as in the commutative case, with the box in the upper right corner and the sign $(-1)^{n-1}$, where n is the order of the quasi-minor. The ribbon Schur functions are the only quasi-Schur functions which are polynomials in the S_k, and they reduce to the ordinary ribbon functions in the commutative case. To emphasize this point, we shall also denote the quasi-Schur function $\check{S}_{I/J}$ by $S_{I/J}$ when I/J is a ribbon.

Quasi-Schur functions are indexed by *partitions*. It would have been possible to define more general functions indexed by compositions, but the homological relations imply that such functions can always be expressed as noncommutative rational fractions in the quasi-Schur functions. For instance

$$\check{S}_{42} = - \begin{vmatrix} S_4 & \boxed{S_3} \\ S_3 & S_2 \end{vmatrix} = - \begin{vmatrix} \boxed{S_3} & S_4 \\ S_2 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & \boxed{S_4} \\ S_2 & S_3 \end{vmatrix} \; S_3^{-1}\, S_2 = \check{S}_{33}\, S_3^{-1}\, S_2 \ .$$

Definition 3.17 is a noncommutative analog of the so-called Jacobi-Trudi formula. Using Jacobi's theorem for the quasi-minors of the inverse matrix as in the proof of Proposition 3.15, we derive the following analog of Naegelbasch's formula.

Proposition 3.18 *Let I be a partition and let $I^\sim = (j_1, \ldots, j_p)$ be its conjugate partition, i.e. the partition whose diagram is obtained by interchanging the rows and columns of the diagram of I. Then,*

$$\check{S}_I = (-1)^{p-1} \begin{vmatrix} \Lambda_{j_p} & \Lambda_{j_p+1} & \cdots & \boxed{\Lambda_{j_p+p-1}} \\ \Lambda_{j_{p-1}-1} & \Lambda_{j_{p-1}} & \cdots & \Lambda_{j_{p-1}+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{j_1-p+1} & \Lambda_{j_1-p+2} & \cdots & \Lambda_{j_1} \end{vmatrix} \ .$$

Let us extend ω to the skew field generated by the S_k. Then we have.

Proposition 3.19 *Let I be a partition and let $I^\sim$ be its conjugate partition. There holds*

$$\omega(\check{S}_I) = \check{S}_{I^\sim} \ .$$

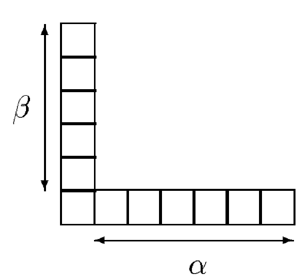
Proof — This is a consequence of Proposition 3.18. □

We can also extend the $*$ involution to the division ring generated by the S_i. It follows from Definition 2.1 that $\check{S}_I^*$ is equal to the transposed quasi-determinant

$$\check{S}_I^* = (-1)^{n-1} \begin{vmatrix} S_{i_n} & S_{i_n+1} & \cdots & \boxed{S_{i_n+n-1}} \\ S_{i_{n-1}-1} & S_{i_{n-1}} & \cdots & S_{i_{n-1}+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i_1-n+1} & S_{i_1-n+2} & \cdots & S_{i_1} \end{vmatrix} \ .$$

In particular, if $I = n^k$ is a rectangular partition, the quasi-Schur function indexed by I is invariant under $*$.

In the commutative case, Schur functions can also be expressed as determinants of hook Schur functions, according to a formula of Giambelli. This formula is stated more conveniently, using Frobenius' notation for partitions. In this notation, the hook



is written $(\beta|\alpha) := 1^\beta\,(\alpha+1)$. A general partition is decomposed into diagonal hooks. Thus, for the partition $I = (2,3,5,7,7)$ for instance, we have the following decomposition, where the different hooks are distinguished by means of the symbols $\star$, $\bullet$ and $\diamond$.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-2}
\star & \bullet \\
\cline{1-3}
\star & \bullet & \diamond \\
\cline{1-5}
\star & \bullet & \diamond & \diamond & \diamond \\
\hline
\star & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\star & \star & \star & \star & \star & \star & \star \\
\hline
\end{array}
$$

It is denoted $(134 \mid 256)$ in Frobenius' notation. We can now state.

Proposition 3.20 (Giambelli's formula for quasi-Schur functions) *Let I be a partition represented by $(\beta_1 \ldots \beta_k \mid \alpha_1 \ldots \alpha_k)$ in Frobenius' notation. One has*

$$
\check{S}_I = \begin{vmatrix}
\check{S}_{(\beta_1|\alpha_1)} & \check{S}_{(\beta_1|\alpha_2)} & \cdots & \check{S}_{(\beta_1|\alpha_k)} \\
\check{S}_{(\beta_2|\alpha_1)} & \check{S}_{(\beta_2|\alpha_2)} & \cdots & \check{S}_{(\beta_2|\alpha_k)} \\
\vdots & \vdots & \ddots & \vdots \\
\check{S}_{(\beta_k|\alpha_1)} & \check{S}_{(\beta_k|\alpha_2)} & \cdots & \boxed{\check{S}_{(\beta_k|\alpha_k)}}
\end{vmatrix} .
$$

Proof — This is an example of relation between the quasi-minors of the matrix $\mathbf{S}$. The proposition is obtained by means of Bazin's theorem for quasi-determinants 2.18. It is sufficient to illustrate the computation in the case of $I = (2,3,5,7,7) = (134 \mid 256)$. We denote by $|i_1 i_2 i_3 i_4 \boxed{i_5}|$ the quasi-minor of $\mathbf{S}$ defined by

$$
|i_1 i_2 i_3 i_4 \boxed{i_5}| = \begin{vmatrix}
S_{i_1} & S_{i_2} & \cdots & \boxed{S_{i_5}} \\
S_{i_1-1} & S_{i_2-1} & \cdots & S_{i_5-1} \\
\vdots & \vdots & \ddots & \vdots \\
S_{i_1-4} & S_{i_2-4} & \cdots & S_{i_5-4}
\end{vmatrix} .
$$

Using this notation, we have

$$
\begin{vmatrix}
\check{S}_{(1|2)} & \check{S}_{(1|5)} & \check{S}_{(1|6)} \\
\check{S}_{(3|2)} & \check{S}_{(3|5)} & \check{S}_{(3|6)} \\
\check{S}_{(4|2)} & \check{S}_{(4|5)} & \boxed{\check{S}_{(4|6)}}
\end{vmatrix}
=
\begin{vmatrix}
|0124\boxed{7}| & |0124\boxed{10}| & |0124\boxed{11}| \\
|0234\boxed{7}| & |0234\boxed{10}| & |0234\boxed{11}| \\
|1234\boxed{7}| & |1234\boxed{10}| & \boxed{|1234\boxed{11}|}
\end{vmatrix} ,
$$

$$
= |\boxed{0}1234| \; |\boxed{0}247\ 10|^{-1} \; |247\ 10\ \boxed{11}| = |247\ 10\ \boxed{11}| = \check{S}_{23577} ,
$$

the second equality following from Bazin's theorem. $\square$

There is also an expression of quasi-Schur functions as quasi-determinants of ribbon Schur functions, which extends to the noncommutative case a formula given in [LP]. To state it, we introduce some notations. A ribbon Θ can be seen as the outer strip of a Young diagram D. The unique box of Θ which is on the diagonal of D is called the *diagonal box* of Θ. The boxes of Θ which are strictly above the diagonal form a ribbon denoted Θ^+. Similarly those strictly under the diagonal form a ribbon denoted Θ^-. Given two ribbons Θ and Ξ, we shall denote by $\Theta^+ \& \,\Xi^-$ the ribbon obtained from Θ by replacing Θ^- by Ξ^-. For example, with Θ and Ξ corresponding to the compositions

$I = (2,1,1,3,2,4)$, $J = (1,3,3,1,1,2)$, Θ^+ corresponds to $(2,1,1,1)$, Ξ^- to $(1,1,1,2)$ and $\Theta^+ \& \Xi^-$ to $(2,1,1,3,1,1,2)$. Given a partition I, we can peel off its diagram into successive ribbons $\Theta_p, \ldots, \Theta_1$, the outer one being Θ_p (see [LP]). Using these notations, we have the following identity.

Proposition 3.21 *Let I be a partition and $(\Theta_p, \ldots, \Theta_1)$ its decomposition into ribbons. Then, we have*

$$\check{S}_I = \begin{vmatrix} \check{S}_{\Theta_1} & \check{S}_{\Theta_1^+ \& \Theta_2^-} & \cdots & \check{S}_{\Theta_1^+ \& \Theta_p^-} \\ \check{S}_{\Theta_2^+ \& \Theta_1^-} & \check{S}_{\Theta_2} & \cdots & \check{S}_{\Theta_2^+ \& \Theta_p^-} \\ \vdots & \vdots & \ddots & \vdots \\ \check{S}_{\Theta_p^+ \& \Theta_1^-} & \check{S}_{\Theta_p^+ \& \Theta_2^-} & \cdots & \boxed{\check{S}_{\Theta_p}} \end{vmatrix} .$$

Proof — The two proofs proposed in [LP] can be adapted to the noncommutative case. The first one, which rests upon Bazin's theorem, is similar to the proof of Giambelli's formula given above. The second one takes Giambelli's formula as a starting point, and proceeds by a step by step deformation of hooks into ribbons, using at each stage the multiplication formula for ribbons and subtraction to a row or a column of a multiple of an other one. To see how it works, let us consider for example the quasi-Schur function $\check{S}_{235}$. Its expression in terms of hooks is

$$\check{S}_{235} = \begin{vmatrix} R_{12} & R_{15} \\ R_{112} & \boxed{R_{115}} \end{vmatrix} .$$

Subtracting to the second row the first one multiplied to the left by R_1 (which does not change the value of the quasi-determinant) and using the multiplication formula for ribbons, we arrive at a second expression

$$\check{S}_{235} = \begin{vmatrix} R_{12} & R_{15} \\ R_{22} & \boxed{R_{25}} \end{vmatrix} .$$

Now, subtracting to the second column the first one multiplied to the right by R_3 we finally obtain

$$\check{S}_{235} = \begin{vmatrix} R_{12} & R_{123} \\ R_{22} & \boxed{R_{223}} \end{vmatrix} ,$$

which is the required expression. □

4 Transition matrices

This section is devoted to the study of the transition matrices between the previously introduced bases of **Sym**. As we shall see, their description is rather simpler than in the commutative case. We recall that **Sym** is a graded algebra

$$\mathbf{Sym} = \bigoplus_{n\geq 0} \mathbf{Sym}_n$$

$\mathbf{Sym}_n$ being the subspace of dimension 2^{n-1} generated by the symmetric functions S^I, for all compositions I of n.

The description of the transition matrices can be abridged by using the involution ω. Note that the action of ω on the usual bases is given by

$$\omega(S^I) = \Lambda^{\overline{I}}, \quad \omega(\Lambda^I) = S^{\overline{I}}, \quad \omega(\Psi^I) = (-1)^{|I|-\ell(I)}\,\Psi^{\overline{I}}, \quad \omega(\Phi^I) = (-1)^{|I|-\ell(I)}\,\Phi^{\overline{I}},$$

where we denote by $\overline{I}$ the mirror image of the composition I, i.e. the new composition obtained by reading I from right to left.

We shall consider two orderings on the set of all compositions of an integer n. The first one is the *reverse refinement order*, denoted $\preceq$, and defined by $I \preceq J$ iff J is finer than I. For example, $(326) \preceq (212312)$. The second one is the reverse lexicographic ordering, denoted $\leq$. For example, $(6) \leq (51) \leq (42) \leq (411)$. This ordering is well suited for the indexation of matrices, thanks to the following property.

Lemma 4.1 *Let C_n denote the sequence of all compositions of n sorted by reverse lexicographic order. Then,*

$$C_n = (\,1 \triangleright C_{n-1}\ ,\ 1\,.\,C_{n-1}\,)$$

where $1 \triangleright C_{n-1}$ and $1\,.\,C_{n-1}$ denote respectively the compositions obtained from the compositions of C_{n-1} by adding 1 to their first part and by considering 1 as their new first part, the other parts being unchanged.

Note 4.2 Writing $S_1^n = S_1\,S_1^{n-1}$, Lemma 4.1 proves in particular that

$$(S_1)^n = \sum_{|I|=n} R_I\ , \tag{56}$$

a noncommutative analogue of a classical formula which is relevant in the representation theory of the symmetric group (the left hand side is the characteristic of the regular representation, which is thus decomposed as the direct sum of all the ribbon representations). This decomposition appears when the regular representation is realized in the space $\mathcal{H}_n$ of $\mathbf{S}_n$-harmonic polynomials, or, which amounts to the same, in the cohomology of the variety of complete flags (more details will be given in Section 5.2).

For every pair (F_I), (G_I) of graded bases of **Sym** indexed by compositions, we denote by $M(F,G)_n$ the transition matrix from the basis (F_I) with $|I| = n$ to the basis (G_I) with $|I| = n$ indexed by compositions sorted in reverse lexicographic order. Our convention is that the *row* indexed by I contains the components of F_I in the basis (G_J).

4.1 S and Λ

The matrices $M(S,\Lambda)_n$ and $M(\Lambda,S)_n$ are easily described. With the choosen indexation, they appear as Kronecker powers of a simple 2×2 matrix.

Proposition 4.3 *For every $n\geq 1$, we have*

$$M(S,\Lambda)_n = M(\Lambda,S)_n = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^{\otimes(n-1)} . \tag{57}$$

Proof — The defining relation $\sigma(-t)\,\lambda(t)=1$ shows that

$$\sigma(-t) = (\,1 - \sum_{i\geq 1} \Lambda_i\, t^i\,)^{-1} = 1 + \sum_{k\geq 1} (-1)^k\,(\,\sum_{i\geq 1} \Lambda_i\, t^i\,)^k \ .$$

Identifying the coefficients, we get

$$S_k = \sum_{|J|=k} (-1)^{l(J)-k}\,\Lambda^J \ ,$$

so that

$$S^I = \sum_{J\succeq I} (-1)^{l(J)-|I|}\,\Lambda^J \ ,$$

for every composition I. The conclusion follows then from Lemma 4.1. Applying ω to this relation, we see that the same relation holds when S and Λ are interchanged. $\square$

Example 4.4 For $n=2$ and $n=3$, we have

$$M(S,\Lambda)_2 = M(\Lambda,S)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} ,$$

$$M(S,\Lambda)_3 = M(\Lambda,S)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

As shown in the proof of Proposition 4.3, we have

$$S^I = \sum_{J\succeq I} (-1)^{\ell(J)-|I|}\,\Lambda^J \ , \quad \Lambda^I = \sum_{J\succeq I} (-1)^{\ell(J)-|I|}\,S^J \ ,$$

for every composition I. Hence the matrices $M(S,\Lambda)_n = M(\Lambda,S)_n$ are upper triangular matrices whose entries are only 0, 1 or -1 and such that moreover the non-zero entries of each column are all equal.

4.2 S and Ψ

Before describing the matrices $M(S,\Psi)_n$ and $M(\Psi,S)_n$, let us introduce some notations. Let $I=(i_1,\dots,i_m)$ be a composition. We define $\pi_u(I)$ as follows

$$\pi_u(I) = i_1\,(i_1+i_2)\,\dots\,(i_1+i_2+\dots+i_m)\ .$$

In other words, $\pi_u(I)$ is the product of the successive partial sums of the entries of the composition I. We also use a special notation for the last part of the composition by setting $lp(I)=i_m$. Let now J be a composition which is finer than I. Let then $J=(J_1,\dots,J_m)$ be the unique decomposition of J into compositions $(J_i)_{i=1,m}$ such that $|J_p|=i_p$, $p=1,\dots,m$. We now define

$$\pi_u(J,I) = \prod_{i=1}^{m} \pi_u(J_i)\ .$$

Similarly, we set

$$lp(J,I) = \prod_{i=1}^{m} lp(J_i)\ ,$$

which is just the product of the last parts of all compositions J_i.

Proposition 4.5 *For every composition I, we have*

$$S^I = \sum_{J\succeq I} \frac{1}{\pi_u(J,I)}\,\Psi^J\ ,$$

$$\Psi^I = \sum_{J\succeq I} (-1)^{\ell(J)-\ell(I)}\ lp(J,I)\ S^J\ .$$

Proof — The two formulas of this proposition are consequences of the quasi-determinantal relations given in Corollary 3.6. Let us establish the first one. According to relation (39) and to basic properties of quasi-determinants, we can write

$$n\,S_n = \begin{vmatrix} \Psi_1 & \Psi_2 & \dots & \Psi_{n-1} & \boxed{\Psi_n} \\ -1 & \Psi_1 & \dots & \Psi_{n-2} & \Psi_{n-1} \\ 0 & -1 & \dots & \dfrac{\Psi_{n-3}}{2} & \dfrac{\Psi_{n-2}}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & \dfrac{\Psi_1}{n+1} \end{vmatrix}\ .$$

But this quasi-determinant can be explicitely expanded by means of Proposition 2.6

$$S_n = \sum_{|J|=n} \frac{1}{\pi_u(J)}\,\Psi^J\ . \tag{58}$$

Taking the product of these identities for $n=i_1,i_2,\dots,i_m$, one obtains the first formula of Proposition 4.5. The second one is established in the same way, starting from relation (41) which expresses Ψ_n as a quasi-determinant in the S_i. $\square$

Note 4.6 One can also prove (58) by solving the differential equation $\sigma'(t) = \sigma(t)\,\psi(t)$ in terms of iterated integrals. This yields

$$\sigma(t) = 1+\int_0^t dt_1\,\psi(t_1)+\int_0^t dt_1\int_0^{t_1} dt_2\,\psi(t_2)\psi(t_1)+\int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\,\psi(t_3)\psi(t_2)\psi(t_1)+\cdots \tag{59}$$

and one obtains relation (58) by equating the coefficients of t^n in both sides of (59).

The matrices $M(\Psi, S)_n$ have a simple block structure.

Proposition 4.7 *For every $n \geq 0$, we have*

$$M(\Psi,S)_n = \begin{pmatrix} M(\Psi,S)_{n-1} + A_{n-1} & -M(\Psi,S)_{n-1} \\ 0 & M(\Psi,S)_{n-1} \end{pmatrix}$$

where A_n is the matrix of size 2^{n-1} defined by

$$A_n = \begin{pmatrix} I_{2^{n-2}} & 0 \\ 0 & M(\Psi,S)_{n-1} \end{pmatrix} ,$$

$M(\Psi,S)_0$ denoting the empty matrix.

Proof — This follows from Proposition 4.5 and Lemma 4.1. □

Example 4.8 For $n = 2$ and $n = 3$, we have

$$M(S,\Psi)_2 = \begin{array}{c|c} & \begin{matrix} 2 & 11 \end{matrix} \\ \begin{matrix} 2 \\ 11 \end{matrix} & \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \end{array} , \quad M(\Psi,S)_2 = \begin{array}{c|c} & \begin{matrix} 2 & 11 \end{matrix} \\ \begin{matrix} 2 \\ 11 \end{matrix} & \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \end{array} ,$$

$$M(S,\Psi)_3 = \begin{array}{c|c} & \begin{matrix} 3 & 21 & 12 & 111 \end{matrix} \\ \begin{matrix} 3 \\ 21 \\ 12 \\ 111 \end{matrix} & \begin{pmatrix} 1/3 & 1/6 & 1/3 & 1/6 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} , \quad M(\Psi,S)_3 = \begin{array}{c|c} & \begin{matrix} 3 & 21 & 12 & 111 \end{matrix} \\ \begin{matrix} 3 \\ 21 \\ 12 \\ 111 \end{matrix} & \begin{pmatrix} 3 & -1 & -2 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

4.3 S and Φ

We first introduce some notations. Let $I = (i_1, \ldots, i_m)$ be a composition and let J be a finer composition. As in the preceding section, decompose $J = (J_1, \ldots, J_m)$ into compositions with respect to the refinement relation $J \succeq I$, *i.e.* such that $|J_p| = i_p$ for every p. Then, we set

$$\ell(J,I) = \prod_{i=1}^{m} \ell(J_i) ,$$

which is just the product of the lengths of the different compositions J_i. We also denote by $\pi(I)$ the product of all parts of I. We also set

$$sp(I) = \ell(I)!\ \pi(I) = m!\ i_1 \ldots i_m .$$

and

$$sp(J,I) = \prod_{i=1}^{m} sp(J_i) .$$

The following proposition can be found in [GaR] and in [Re].

Proposition 4.9 *For every composition I, we have*

$$\Phi^I = \sum_{J \succeq I} (-1)^{\ell(J)-\ell(I)} \frac{\pi(I)}{\ell(J,I)} S^J ,$$

$$S^I = \sum_{J \succeq I} \frac{1}{sp(J,I)} \Phi^J .$$

Proof — Using the series expansion of the logarithm and the defining relation (26) yields

$$\Phi_n = \sum_{|J|=n} (-1)^{\ell(J)-1} \frac{n}{\ell(J)} S^J ,$$

from which the first part of the proposition follows. On the other hand, the series expansion of the exponential and the defining relation (25) gives

$$S_n = \sum_{|J|=n} \frac{1}{sp(J)} \Phi^J ,$$

which implies the second equality. □

Note 4.10 It follows from Proposition 4.9 that the coefficients of the expansion of S_n on the basis Φ^I only depend on the partition $\sigma(I) = (1^{m_1} 2^{m_2} \ldots n^{m_n})$ associated with I. In fact, one has

$$sp(I) = \binom{m_1 + \ldots + m_n}{m_1, \ \ldots \ , m_n} \ 1^{m_1} \ldots n^{m_n} \ m_1! \ \ldots \ m_n! ,$$

which shows that $1/sp(I)$ is equal to the usual commutative scalar product $(S_n \, , \, \psi_{\sigma(I)})$ divided by the number of compositions with associated partition $\sigma(I)$.

Example 4.11 For $n = 2$ and $n = 3$, we have

$$M(S,\Phi)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \end{array} , \quad M(\Phi,S)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \end{array} ,$$

$$M(S,\Phi)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1/3 & 1/4 & 1/4 & 1/6 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} , \quad M(\Phi,S)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 3 & -3/2 & -3/2 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

Note 4.12 One can produce combinatorial identities by taking the commutative images of the relations between the various bases of noncommutative symmetric functions. For example, taking into account the fact that Ψ and Φ reduce in the commutative case to the same functions, one deduces from the description of the matrices $M(S,\Psi)$ and $M(S,\Phi)$ that

$$\sum_{\sigma(J)=I} \frac{1}{\pi_u(J)} = \frac{1}{\pi(I)} ,$$

the sum in the left-hand side being taken over all compositions corresponding to the same partition I.

4.4 S and R

The matrices $M(S,R)_n$ and $M(R,S)_n$ are given by Kronecker powers of 2×2 matrices.

Proposition 4.13 *For every $n \geq 1$, we have*

$$M(S,R)_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\otimes(n-1)} , \tag{60}$$

$$M(R,S)_n = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{\otimes(n-1)} . \tag{61}$$

Proof — It is sufficient to establish the second relation, since one has

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} .$$

Going back to the definition of a ribbon Schur function and using the same technique as in the proof of Proposition 4.5, one arrives at

$$R_I = \sum_{I \succeq J} (-1)^{\ell(J)-\ell(I)}\, S^J ,$$

for every composition I. The conclusion follows again from Lemma 4.1. $\square$

Example 4.14 For $n = 2$ and $n = 3$, we have

$$M(S,R)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array} , \quad M(R,S)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array} ,$$

$$M(S,R)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} , \quad M(R,S)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{array} .$$

It follows from the proof of Proposition 4.13 that

$$S^I = \sum_{I \succeq J} R_J , \quad R_I = \sum_{I \succeq J} (-1)^{\ell(I)-\ell(J)}\, S^J , \tag{62}$$

for every composition I. This is the noncommutative analog of a formula of MacMahon (*cf.* [MM]). These formulas are equivalent to the well-known fact that the Möbius function of the order $\preceq$ on compositions is equal to $\mu_{\preceq}(I,J) = (-1)^{\ell(I)-\ell(J)}$ if $I \succeq J$ and to $\mu_{\preceq}(I,J) = 0$ if $I \prec J$.

4.5 Λ and Ψ

The matrices relating the two bases Λ and Ψ can be described similarly to the matrices relating S and Ψ with the only difference that the combinatorial descriptions reverse left and right.

Proposition 4.15 *For every composition I, we have*

$$\Lambda^I = \sum_{J \succeq I} (-1)^{|I|-\ell(J)} \frac{1}{\pi_u(\overline{J},\overline{I})} \Psi^J ,$$

$$\Psi^I = (-1)^{|I|} \sum_{J \succeq I} (-1)^{\ell(J)} \, lp(\overline{J},\overline{I}) \, \Lambda^J .$$

Proof — It suffices to apply ω to the formulas given by Proposition 4.5. □

The block structure of $M(\Psi,\Lambda)_n$ is also simple.

Proposition 4.16 *For $n \geq 2$,*

$$M(\Psi,\Lambda)_n = \begin{pmatrix} -A_{n-1} - M(\Psi,\Lambda)_{n-1} & A_{n-1} \\ 0 & M(\Psi,\Lambda)_{n-1} \end{pmatrix}$$

where A_n is the matrix of size 2^{n-1} defined by

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ 0 & M(\Psi,\Lambda)_{n-1} \end{pmatrix} ,$$

with $A_1 = (1)$ and $M_1 = (1)$.

Proof — This follows from Propositions 4.7 and 4.3 using the fact that

$$M(\Psi,\Lambda)_n = M(\Psi,S)_n \, M(S,\Lambda)_n .$$

□

Example 4.17 For $n = 2$ and $n = 3$, one has

$$M(\Psi,\Lambda)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} \end{array} , \quad M(\Lambda,\Psi)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} -1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \end{array} ,$$

$$M(\Psi,\Lambda)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 3 & -2 & -1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} , \quad M(\Lambda,\Psi)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1/3 & -1/3 & -1/6 & 1/6 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

4.6 Λ and Φ

The matrices relating Λ and Φ are again, due to the action of ω, essentially the same as the matrices relating S and Φ.

Proposition 4.18 *For every composition I, we have*

$$\Phi^I = \sum_{J \succeq I} (-1)^{|I|-\ell(J)} \frac{\pi(I)}{\ell(J,I)} \Lambda^J ,$$

$$\Lambda^I = \sum_{J \succeq I} (-1)^{|I|-\ell(J)} \frac{1}{sp(J,I)} \Phi^J .$$

Proof — It suffices again to apply ω to the relations given by Proposition 4.9. $\square$

Example 4.19 For $n = 2$ and $n = 3$, we have

$$M(\Lambda,\Phi)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} -1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \end{array} , \quad M(\Phi,\Lambda)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} \end{array} ,$$

$$M(\Phi,\Lambda)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 3 & -3/2 & -3/2 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} , \quad M(\Lambda,\Phi)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 1/3 & -1/4 & -1/4 & 1/6 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

4.7 Λ and R

The matrices $M(\Lambda,R)$ and $M(R,\Lambda)$ are again Kronecker powers of simple 2×2 matrices.

Proposition 4.20 *For every $n \geq 1$, we have*

$$M(\Lambda,R)_n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{\otimes(n-1)} , \quad M(R,\Lambda)_n = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes(n-1)} .$$

Proof — This follows from Propositions 4.13 and 4.3, since

$$M(\Lambda,R)_n = M(\Lambda,S)_n \, M(\mathbf{S},R)_n \quad \text{and} \quad M(R,\Lambda)_n = M(R,S)_n \, M(S,\Lambda)_n .$$

Another possibility is to apply ω to the formulas (62). $\square$

Note 4.21 Proposition 4.20 is equivalent to the following formulas

$$\Lambda^I = \sum_{\overline{I} \succeq J^\sim} R_J , \quad R_I = \sum_{I^\sim \succeq \overline{J}} (-1)^{\ell(I^\sim)-\ell(J)} \Lambda^J . \tag{63}$$

Example 4.22 For $n = 2$ and $n = 3$, we have

$$M(\Lambda,R)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} , \quad M(R,\Lambda)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \end{array} ,$$

$$M(\Lambda,R)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} , \quad M(R,\Lambda)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} .$$

4.8 Ψ and R

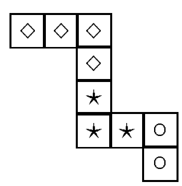
Let $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ be two compositions of n. The ribbon decomposition of I relatively to J is the unique decomposition of I as

$$I = I_1 \bullet I_2 \bullet \ \ldots \ \bullet I_s \ , \tag{64}$$

where I_i denotes a composition of length j_i for every i and where $\bullet$ stands for $\cdot$ or $\triangleright$ as defined in the statement of Proposition 3.13. For example, if $I = (3,1,1,3,1)$ and $J = (4,3,2)$, we have

$$I = (31) \cdot (12) \triangleright (11) \ ,$$

which can be read on the ribbon diagram associated with I:



For a pair I, J of compositions of the same integer n, let $(I_1, \ldots, I_s)$ be the ribbon decomposition of I relatively to J as given by (64). We define then $psr(I,J)$ by

$$psr(I,J) = \begin{cases} (-1)^{l(I_1)+\ldots+l(I_s)-s} & \text{if every } I_j \text{ is a hook} \\ 0 & \text{otherwise} \end{cases} .$$

We can now give the expression of Ψ^I in the basis of ribbon Schur functions.

Proposition 4.23 *For any composition I,*

$$\Psi^I = \sum_{|J|=n} psr(J,I)\, R_J \ . \tag{65}$$

Proof — This follows from Corollary 3.14 and Proposition 3.13. □

The block structure of the matrix $M(\Psi,R)_n$ can also be described.

Proposition 4.24

$$M(\Psi,R)_n = \begin{pmatrix} A_{n-1} & -M(\Psi,R)_{n-1} \\ M(\Psi,R)_{n-1} & M(\Psi,R)_{n-1} \end{pmatrix} ,$$

where A_n is a matrix of order 2^{n-1}, itself given by the block decomposition

$$A_n = \begin{pmatrix} A_{n-1} & 0 \\ M(\Psi,R)_{n-1} & M(\Psi,R)_{n-1} \end{pmatrix} .$$

Proof — The result follows from Proposition 4.23 and from Lemma 4.1. □

The structure of the matrix $M(R,\Psi)_n$ is more complicated. To describe it, we introduce some notations. For a vector $v = (v_1, \ldots, v_n) \in \mathbf{Q}^n$, we denote by $v[i,j]$ the vector $(v_i, v_{i+1}, \ldots, v_j) \in \mathbf{Q}^{j-i}$. We also denote by $\overline{v}$ the vector obtained by reading the entries of v from right to left and by $v \,.\, w$ the vector obtained by concatenating the entries of v and w.

Proposition 4.25 *1) One can write $M(R,\Psi)_n$ in the following way*

$$M(R,\Psi)_n = \frac{1}{n!} \begin{pmatrix} A(R,\Psi)_n & A(R,\Psi)_n \\ -A(R,\Psi)_n & B(R,\Psi)_n \end{pmatrix}$$

where $A(R,\Psi)_n$ and $B(R,\Psi)_n$ are matrices of order 2^{n-2} whose all entries are integers and that satisfy to the relation

$$A(R,\Psi)_n + B(R,\Psi)_n = n\, M(R,\Psi)_{n-1} .$$

Hence $M(R,\Psi)_n$ is in particular completely determined by the structure of $B(R,\Psi)_n$.

2) The matrix $B(R,\Phi)_n$ can be block decomposed as follows

$$B(R,\Phi)_n = \left(\begin{array}{cccc} & & -B_n^{(000)} & \ldots \\ & -B_n^{(00)} & B_n^{(000)} & \ldots \\ -B_n^{(0)} & & -B_n^{(001)} & \ldots \\ & B_n^{(00)} & B_n^{(001)} & \ldots \\ & & -B_n^{(010)} & \ldots \\ & -B_n^{(01)} & B_n^{(010)} & \ldots \\ B_n^{(0)} & & -B_n^{(011)} & \ldots \\ & B_n^{(01)} & B_n^{(011)} & \ldots \end{array} \right) \qquad (66)$$

where $(B_n^{(i_1,\ldots,i_r)})_{r=1,\ldots,n-1}$ are square blocks of order 2^{n-1-r}.

3) Every block $B_n^{(i_1,\ldots,i_r)}$ has itself a block structure of the form given by (66) with blocks denoted here $B_{(i_1,\ldots,i_n)}^{(j_1,\ldots,j_s)}$. These blocks statisfy the two following properties.

– *Every block $B_{(i_1,\ldots,i_n)}^{(j_1,\ldots,j_s)}$ of order 2×2 has the structure*

$$B_{(i_1,\ldots,i_n)}^{(j_1,\ldots,j_s)} = \begin{pmatrix} p & q \\ -p & p \end{pmatrix} ,$$

for some integers $p, q \in \mathbf{Z}$.

– *For every* $(i_1, \ldots, i_n)$ *and* $(j_1, \ldots, j_s)$, *let us consider the rectangular matrix* $C_{(i_1,\ldots,i_r)}^{(j_1,\ldots,j_s)}$ *defined as follows*

$$C_{(i_1,\ldots,i_r)}^{(j_1,\ldots,j_s)} = \begin{pmatrix} B_{(i_1,\ldots,i_n)}^{(j_1,\ldots,j_s)} & \begin{matrix} -B_{(i_1,\ldots,i_n)}^{(j_1,\ldots,j_s,0)} \\ B_{(i_1,\ldots,i_n)}^{(j_1,\ldots,j_s,0)} \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} \end{pmatrix} .$$

If we denote by $LC(M)$ *the last column of a matrix* M, *we then have*

$$LC(B_{(i_1,\ldots,i_r)}^{(j_1,\ldots,j_s)}) = (\, LC(-B_{(i_1,\ldots,i_r)}^{(j_1,\ldots,j_s,0)}) \cdot LC(B_{(i_1,\ldots,i_r)}^{(j_1,\ldots,j_s,0)})\,) \; - \; LC(C_{(i_1,\ldots,i_r)}^{(j_1,\ldots,j_s)}) \; .$$

Note that these recursive properties allow to recover all the block structure of $B(R,\Psi)_n$ *from the last column of this matrix.*

4) Let V_n *be the vector of order* 2^{n-2} *corresponding to the last column of* $B(R,\Psi)_n$ *read from* bottom to top. *The vector* V_n *is then determined by the recursive relations*

$$V_2 = (\,1\,) \; ,$$

$$V_n[1] = 1 \; , \quad V_n[2] = n-1 \; ,$$

$$V_n[1, 2^k] + V_n[2^k+1, 2^{k+1}] = \begin{pmatrix} n \\ k+1 \end{pmatrix} \; V_k[1, 2^{k-1}] \,.\, \overline{V_k[1, 2^{k-1}]} \; ,$$

for $k \in \{1, \ldots, n-4\}$, *and*

$$V_n[1, 2^{n-3}] + \overline{V_n[2^{n-3}+1, 2^{n-2}]} = n \;\, V_{n-1} \; .$$

Example 4.26 Here are the first vectors V_n from which the matrix $M(R,\Psi)_n$ can be recovered:

$$V_2 = (\,1\,) \; , \; V_3 = (\,1 \;\; 2\,) \; , \; V_4 = (\,1 \;\; 3 \;\; 5 \;\; 3\,) \; , \; V_5 = (\,1 \;\; 4 \;\; 9 \;\; 6 \;\; 9 \;\; 16 \;\; 11 \;\; 4\,) \; ,$$

$$V_6 = (\,1 \;\; 5 \;\; 14 \;\; 10 \;\; 19 \;\; 35 \;\; 26 \;\; 10 \;\; 14 \;\; 40 \;\; 61 \;\; 35 \;\; 26 \;\; 40 \;\; 19 \;\; 5\,) \; .$$

For $n = 2, 3, 4$ the matrices relating Ψ^I and R_I.

$$M(\Psi,R)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{array} \; , \quad M(R,\Psi)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \end{array} \; ,$$

$$M(\Psi,R)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \; , \quad M(R,\Psi)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 1/3 & 1/6 & 1/3 & 1/6 \\ -1/3 & 1/3 & -1/3 & 1/3 \\ -1/3 & -1/6 & 1/6 & 1/3 \\ 1/3 & -1/3 & -1/6 & 1/6 \end{pmatrix} \end{array} \; ,$$

$$M(\Psi,R)_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} ,$$

$$M(R,\Psi)_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\ \begin{pmatrix} 1/4 & 1/12 & 1/8 & 1/24 & 1/4 & 1/12 & 1/8 & 1/24 \\ -1/4 & 1/4 & -1/8 & 1/8 & -1/4 & 1/4 & -1/8 & 1/8 \\ -1/4 & -1/12 & 1/8 & 5/24 & -1/4 & -1/12 & 1/8 & 5/24 \\ 1/4 & -1/4 & -1/8 & 1/8 & 1/4 & -1/4 & -1/8 & 1/8 \\ -1/4 & -1/12 & -1/8 & -1/24 & 1/12 & 1/12 & 5/24 & 1/8 \\ 1/4 & -1/4 & 1/8 & -1/8 & -1/12 & 1/12 & -5/24 & 5/24 \\ 1/4 & 1/12 & -1/8 & -5/24 & -1/12 & -1/12 & 1/24 & 1/8 \\ -1/4 & 1/4 & 1/8 & -1/8 & 1/12 & -1/12 & -1/24 & 1/24 \end{pmatrix} \end{array} .$$

4.9 Φ and R

The first row of $M(\Phi,R)_n$ is given by the following formula, which is equivalent to Corollary 3.16 of [Re] p. 42.

Proposition 4.27 *The expansion of Φ_n in the basis (R_I) is given by*

$$\Phi_n = \sum_{|I|=n} \frac{(-1)^{\ell(I)-1}}{\binom{n-1}{\ell(I)-1}} \, R_I \ .$$

Let I, J be two compositions of the same integer n and let $I = (I_1, \ldots, I_s)$ be the ribbon decomposition of I relatively to J as given by relation (64). Define $phr(I,J)$ by setting

$$phr(I,J) = \prod_{i=1}^{s} \frac{(-1)^{\ell(I_i)-1}}{\binom{|I_i|-1}{\ell(I_i)-1}} \ .$$

We can now give the expression of Φ^I in the basis of ribbon Schur functions.

Corollary 4.28 *For every composition I, one has*

$$\Phi^I = \sum_{|J|=n} phr(J,I) \ R_J \ . \qquad (67)$$

Proof — This is a simple consequence of Propositions 4.27 and 3.13. □

On the other hand, the structure of the matrix $M(R,\Phi)_n$ is more intricate.

Proposition 4.29 *1) Let $D(S_n,\Phi)$ be the diagonal matrix constructed with the elements of the first row of $M(R,\Phi)_n$, i.e. the diagonal matrix whose entry of index (I,I) is $1/sp(I)$ according to Proposition 4.9. Then we can write in a unique way*

$$M(R,\Phi)_n = N(R,\Phi)_n \; D(S_n,\Phi)$$

where $N(R,\Phi)_n$ is a matrix of order 2^{n-1} whose all entries are integers.

2) The matrix $N(R,\Phi)_n$ can be block decomposed as follows

$$N(R,\Phi)_n = \left(\begin{array}{cccc}
 & & A_n^{(000)} & \dots \\
 & A_n^{(00)} & & \\
 & & -A_n^{(000)} & \dots \\
A_n^{(0)} & & & \\
 & & A_n^{(001)} & \dots \\
 & -A_n^{(00)} & & \\
 & & -A_n^{(001)} & \dots \\
 & & & \\
 & & A_n^{(010)} & \dots \\
 & A_n^{(01)} & & \\
 & & -A_n^{(010)} & \dots \\
-A_n^{(0)} & & & \\
 & & A_n^{(011)} & \dots \\
 & -A_n^{(01)} & & \\
 & & -A_n^{(011)} & \dots
\end{array}\right)$$

where the blocks $(A_n^{(i_1,\dots,i_r)})_{r=1,\dots,n-1}$ are of order 2^{n-1-r}. Moreover $A_n^{(1)} = N(R,\Phi)_{n-1}$ and

$$A_n^{(i_1,\dots,i_r)} = A_{n-1}^{(i_2,\dots,i_r)} \; ,$$

for every $r \in \{1,\dots,n-2\}$. This shows in particular that the matrix $N(R,\Phi)_n$ is completely determined by its last column.

3) Let $LC(R,\Phi)_n$ be the row vector of order 2^{n-1} which corresponds to the reading of the last column of $N(R,\Phi)_n$ from top to bottom and let V_n be the vector of order 2^{n-2} defined in the statement of Proposition 4.25. Then, one has

$$LC(R,\Phi)_n = V_n \,.\, \overline{V_n} \; .$$

Thus $LC(R,\Phi)_n$, and hence all the matrix $M(R,\Phi)_n$, can be recovered from V_n.

Example 4.30 The matrices relating R_I and Φ^I for $n = 2,3,4$ are given below.

$$M(\Phi,R)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{array} , \quad M(R,\Phi)_2 = \begin{array}{c} \\ 2 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \end{array} ,$$

$$M(\Phi,R)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1 & -1/2 & -1/2 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} , \quad M(R,\Phi)_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 1/3 & 1/4 & 1/4 & 1/6 \\ -1/3 & 1/4 & -1/4 & 1/3 \\ -1/3 & -1/4 & 1/4 & 1/3 \\ 1/3 & -1/4 & -1/4 & 1/6 \end{pmatrix} \end{array} ,$$

$$M(\Phi,R)_4 = \begin{array}{cc} & \begin{matrix} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{matrix} \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -1/3 & -1/3 & 1/3 & -1/3 & 1/3 & 1/3 & -1 \\ 1 & 1 & -1/2 & -1/2 & -1/2 & -1/2 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1/2 & -1/2 & 1 & 1 & -1/2 & -1/2 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} ,$$

$$M(R,\Phi)_4 = \begin{array}{cc} & \begin{matrix} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{matrix} \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1/4 & 1/6 & 1/8 & 1/12 & 1/6 & 1/12 & 1/12 & 1/24 \\ -1/4 & 1/6 & -1/8 & 1/6 & -1/6 & 1/6 & -1/12 & 1/8 \\ -1/4 & -1/6 & 1/8 & 1/6 & -1/6 & -1/12 & 1/6 & 5/24 \\ 1/4 & -1/6 & -1/8 & 1/12 & 1/6 & -1/6 & -1/6 & 1/8 \\ -1/4 & -1/6 & -1/8 & -1/12 & 1/6 & 1/6 & 1/6 & 1/8 \\ 1/4 & -1/6 & 1/8 & -1/6 & -1/6 & 1/12 & -1/6 & 5/24 \\ 1/4 & 1/6 & -1/8 & -1/6 & -1/6 & -1/6 & 1/12 & 1/8 \\ -1/4 & 1/6 & 1/8 & -1/12 & 1/6 & -1/12 & -1/12 & 1/24 \end{pmatrix} \end{array} .$$

4.10 Φ and Ψ

The problem of expressing Φ_n as a linear combination of the Ψ^I is a classical question of Mathematical Physics, known as the problem of the *continuous Baker-Campbell-Hausdorff exponents* [Mag][Wil][BMP][MP]. The usual formulation of this problem is as follows. Let $H(t)$ be a differentiable function with values in some operator algebra, and consider the solution $E(t;t_0)$ (the *evolution operator*) of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}E(t;t_0) = H(t)E(t;t_0) \\ E(t_0;t_0) = 1 \end{cases}$$

The problem is to express the *continuous BCH exponent* $\Omega(t;t_0)$, defined by $E(t;t_0) = \exp\Omega(t;t_0)$ in terms of $H(t)$.

Here we merely consider the (equivalent) variant $\dfrac{\partial E}{\partial t} = E(t)H(t)$ with $t_0 = 0$, and we set $E(t) = \sigma(t)$, $H(t) = \psi(t)$, $\Omega(t) = \Phi(t)$.

The first answer to this question has been given by W. Magnus [Mag]. His formula provides an implicit relation between $\Phi(t)$ and $\psi(t)$, allowing for the recursive computation of the coefficients Φ_n. ¿From our point of view, it rather gives the explicit expression of Ψ_n in terms of the Φ^I. Following [Wil], we shall derive this formula from an identity of independent interest:

Lemma 4.31 *The derivative of $\sigma(t) = \exp\Phi(t)$ is given by*

$$\sigma'(t) = \frac{d}{dt}e^{\Phi(t)} = \int_0^1 e^{(1-u)\Phi(t)}\frac{d\Phi(t)}{dt}e^{u\Phi(t)}du \ . \qquad (68)$$

Proof — Expanding $\sigma(t) = e^{\Phi(t)}$ in powers of $\Phi(t)$, we have

$$\sigma'(t) = \frac{d}{dt}\sum_{n\geq 0}\frac{1}{n!}\Phi(t)^n = \sum_{r,s\geq 0}\frac{\Phi(t)^r\Phi'(t)\Phi(t)^s}{(r+s+1)!}$$

$$= \sum_{r,s\geq 0}\frac{r!s!}{(r+s+1)!}\frac{\Phi(t)^r}{r!}\Phi'(t)\frac{\Phi(t)^s}{s!}\ ,$$

and using the integral representation

$$\frac{r!s!}{(r+s+1)!} = \mathrm{B}(r+1,s+1) = \int_0^1 (1-u)^r u^s du$$

we find

$$\sigma'(t) = \sum_{r,s\geq 0}\int_0^1 \frac{[(1-u)\Phi(t)]^r}{r!}\Phi'(t)\frac{[u\Phi(t)]^s}{s!}du$$

$$= \int_0^1 e^{(1-u)\Phi(t)}\Phi'(t)e^{u\Phi(t)}du\ .$$

□

To obtain an expression for Ψ_n, we observe that Lemma 4.31 can be rewritten

$$\sigma'(t) = \sigma(t)\ \int_0^1 e^{-u\Phi(t)}\Phi'(t)e^{u\Phi(t)}du\ ,$$

so that

$$\psi(t) = \int_0^1 e^{-u\Phi(t)}\Phi'(t)e^{u\Phi(t)}du \tag{69}$$

$$= \int_0^1 \sum_{r\geq 0}\frac{(-u)^r}{r!}\Phi(t)^r\Phi'(t)\sum_{s\geq 0}\frac{u^s}{s!}\Phi(t)^s\, du$$

$$= \sum_{r,s\geq 0}\frac{(-1)^r}{(r+s+1)!}\binom{r+s}{r}\Phi(t)^r\Phi'(t)\Phi(t)^s\ .$$

Extracting the coefficient of t^{n-1}, we finally obtain:

Proposition 4.32 *The expansion of Ψ_n in the basis (Φ^K) is given by*

$$\Psi_n = \sum_{|K|=n}\left[\sum_{i=1}^{\ell(K)}(-1)^{i-1}\binom{\ell(K)-1}{i-1}k_i\right]\frac{\Phi^K}{\ell(K)!\pi(K)}\ .$$

□

Using the symbolic notation

$$\{\Phi_{i_1}\cdots\Phi_{i_r}\ ,\ F\} = \mathrm{ad}\,\Phi_{i_1}\mathrm{ad}\,\Phi_{i_2}\cdots\mathrm{ad}\,\Phi_{i_r}(F) = [\Phi_{i_1},[\Phi_{i_2},[\ldots[\Phi_{i_r}\ ,\ F]\ldots]]]$$

and the classical identity

$$e^a b e^{-a} = \sum_{n\geq 0}\frac{(\mathrm{ad}\,a)^n}{n!}\,b = \{e^a\ ,\ b\}\ ,$$

we can rewrite (69) as

$$\psi(t) = \sum_{n\geq 0} \frac{(-1)^n}{(n+1)!}\{\Phi(t)^n\,,\,\Phi'(t)\} = \left\{\frac{1-e^{-\Phi(t)}}{\Phi(t)}\,,\,\Phi'(t)\right\}$$

which by inversion gives the Magnus formula:

$$\Phi'(t) = \left\{\frac{\Phi(t)}{1-e^{-\Phi(t)}}\,,\,\psi(t)\right\} = \sum_{n\geq 0}\frac{B_n}{n!}(\mathrm{ad}\,\Phi(t))^n\,\psi(t) \tag{70}$$

the B_n being the Bernoulli numbers.

As already mentioned, formula (70) permits the recursive computation of the Φ_n in terms of iterated integrals. There exists, however, an explicit expression of this type, which is due to Bialynicki-Birula, Mielnik and Plebański [BMP] (see also [MP]). We shall here only state a version of this result, postponing the discussion to section 5.5, where the formalism of the internal product will be needed.

Recall that an index $i \in \{1,2,\ldots,n-1\}$ is said to be a *descent* of a permutation $\sigma \in \mathbf{S}_n$ if $\sigma(i) > \sigma(i+1)$. We denote by $d(\sigma)$ the number of descents of the permutation σ.

Theorem 4.33 (Continuous BCH formula) *The expansion of $\Phi(t)$ in the basis (Ψ^I) is given by the series*

$$\Phi(t) = \sum_{r\geq 1}\int_0^t dt_1 \cdots \int_0^{t_{r-1}} dt_r \sum_{\sigma\in\mathbf{S}_r} \frac{(-1)^{d(\sigma)}}{r}\binom{r-1}{d(\sigma)}^{-1} \psi(t_{\sigma(r)})\cdots\psi(t_{\sigma(1)})\,. \tag{71}$$

Thus, the coefficient of $\Psi^I = \Psi_{i_1}\cdots\Psi_{i_r}$ in the expansion of Φ_n is equal to

$$n\int_0^1 dt_1\cdots\int_0^{t_{r-1}} dt_r \sum_{\sigma\in\mathbf{S}_r}\frac{(-1)^{d(\sigma)}}{r}\binom{r-1}{d(\sigma)}^{-1} t_{\sigma(r)}^{i_1-1}\cdots t_{\sigma(1)}^{i_r-1}\,.$$

Example 4.34 Here are the transition matrices corresponding to the two bases Ψ and Φ, up to the order $n=4$

$$M(\Psi,\Phi)_2 = M(\Phi,\Psi)_2 = \begin{array}{c c} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\,,$$

$$M(\Psi,\Phi)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 1 & 1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}\,,\quad M(\Phi,\Psi)_3 = \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 1 & -1/4 & 1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}\,,$$

$$M(\Psi,\Phi)_4 = \begin{array}{c c} & \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\ \begin{array}{c} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} & \begin{pmatrix} 1 & 1/3 & 0 & 1/12 & -1/3 & -1/6 & 1/12 & 0 \\ 0 & 1 & 0 & 1/4 & 0 & -1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/4 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}\,,$$

$$M(\Phi,\Psi)_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\ \begin{pmatrix} 1 & -1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

5 Connections with Solomon's descent algebra

Let us denote by $\mathrm{Des}\,(\sigma)$ the descent set of a permutation σ, and for each subset $A \subseteq \{1,\ldots,n-1\}$, consider the element

$$D_{=A} = \sum_{\mathrm{Des}\,(\sigma)=A} \sigma \quad \in K[S_n] \ .$$

It has been shown by Solomon (*cf.* [So]) that the subspace of $K[S_n]$ generated by the elements $D_{=A}$ is in fact a subalgebra of $K[S_n]$. More precisely, there exist nonnegative integers d_{AB}^C such that

$$D_{=A}\, D_{=B} = \sum_C d_{AB}^C\, D_{=C} \ ,$$

for $A, B \subset \{1,\ldots,n-1\}$. This subalgebra is called the *descent algebra* of S_n and is denoted by Σ_n. A similar algebra can be defined for any Coxeter group (*cf.* [So]).

The dimension of Σ_n as a vector space is obviously 2^{n-1}, that is, the same as the dimension of the space $\mathbf{Sym}_n$ of noncommutative formal symmetric functions of weight n. This is not a mere coincidence : there is a canonical isomorphism between these two spaces, and we shall see below that the whole algebra $\mathbf{Sym}$ can be identified with the direct sum $\Sigma = \bigoplus_{n\geq 0} \Sigma_n$ of all descent algebras, endowed with some natural operations.

5.1 Internal product

Subsets of $[n-1] = \{1,\ldots,n-1\}$ can be represented by compositions of n in the following way. To a subset $A = \{a_1 < a_2 < \ldots < a_k\}$ of $[n-1]$, one associates the composition

$$c(A) = I = (i_1,\ldots,i_{k+1}) = (a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k) \ .$$

A permutation compatible with a ribbon diagram of associated composition I (*i.e.* yielding a standard skew tableau when written in the diagram) has then for descent set $c^{-1}(I)$. It is thus natural to define a linear isomorphism $\alpha : \Sigma_n \longrightarrow \mathbf{Sym}_n$ by setting

$$\alpha(D_{=A}) = R_{c(A)} \ .$$

This allows us to define an algebra structure on each homogeneous component $\mathbf{Sym}_n$ of $\mathbf{Sym}$ by transporting the product of Σ_n. More precisely, we shall use the *opposite* structure, for reasons that will become apparent later. The so-defined product will be denoted by $*$. Hence, for $F, G \in \mathbf{Sym}_n$, we have

$$F * G = \alpha(\alpha^{-1}(G)\, \alpha^{-1}(F)) \ .$$

We then extend this product to $\mathbf{Sym}$ by setting $F * G = 0$ if F and G are homogeneous of different weights. The operation $*$ will be called the *internal product* of $\mathbf{Sym}$, by analogy with the commutative operation of which it is the natural noncommutative analog. Indeed, let us consider another natural basis of Σ_n, constituted by the elements

$$D_{\subseteq A} = \sum_{B \subseteq A} D_{=B} \ .$$

Then, according to formula (62),

$$\alpha(D_{\subseteq A}) = S^I \ ,$$

where $I = c(A)$ (see also [Re]). Remark that in particular

$$\alpha(id) = S_n$$

so that for $F \in \mathbf{Sym}_n$,

$$F * S_n = S_n * F = F \ ,$$

and that

$$\alpha(\omega_n) = \Lambda_n \ ,$$

where ω_n denotes the maximal permutation $(n\, n-1 \cdots 1) = D_{=[1,n-1]}$ of S_n. Finally the multiplication formula for $D_{\subseteq A}\, D_{\subseteq B}$ (*cf.* [So]) can be be rewritten as follows.

Proposition 5.1 *For any two compositions $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$, one has*

$$S^I * S^J = \sum_{M \in \mathrm{Mat}(I,J)} S^M \tag{72}$$

where $\mathrm{Mat}(I, J)$ *denotes the set of matrices of nonnegative integers $M = (m_{ij})$ of order $p \times q$ such that $\sum_s m_{rs} = i_r$ and $\sum_r m_{rs} = j_s$ for $r \in [1, p]$ and $s \in [1, q]$, and where*

$$S^M = S_{m_{11}}\, S_{m_{12}} \cdots S_{m_{1p}} \cdots S_{m_{q1}} \cdots S_{m_{qp}} \ .$$

It is well known that the same formula holds for *commutative* symmetric functions. In this case, a product of complete functions S^I is the Frobenius characteristic of a permutation representation (see [JK]). Thus the passage to commutative symmetric functions transforms the noncommutative $*$-product into the ordinary internal product (this is not surprising, since the descent algebra was originally introduced as a noncommutative analog of the character ring in the group algebra).

It is thus natural to identify **Sym** with the direct sum $\Sigma = \bigoplus_{n \geq 0} \Sigma_n$ of all descent algebras. The ordinary product of **Sym** then corresponds to a natural product on Σ, which we will call *outer product.* In fact Σ can be interpreted as a subalgebra of the convolution algebra of a free associative algebra (considered as a Hopf algebra for the standard comultiplication making the letters primitive). More precisely, it is the subalgebra generated by the projectors q_n onto the homogeneous components $K_n\langle A\rangle$ of the free algebra $K\langle A\rangle$ (*cf.* [Re]). The convolution product, denoted $*$ in [Re], corresponds then to the ordinary product of noncommutative symmetric functions, and the composition of endomorphisms to the internal product. This construction has been recently extended to the case of any graded bialgebra by Patras (*cf.* [Pa]).

The noncommutative internal product satisfy some compatibility relations with the ordinary product, similar to those encountered in the commutative case. For example one has the following identity, whose commutative version is a fundamental tool for calculations with tensor products of symmetric group representations (see [Ro], [Li2] or [Th]).

Proposition 5.2 *Let $F_1, F_2, \ldots, F_r, G \in$ **Sym**. Then,*

$$(F_1 F_2 \cdots F_r) * G = \mu_r \left[(F_1 \otimes \cdots \otimes F_r) * \Delta^r G \right]$$

where in the right-hand side, μ_r denotes the r-fold ordinary multiplication and $*$ stands for the operation induced on $\mathbf{Sym}^{\otimes n}$ by $*$.

Proof — The formula being linear in each of its arguments, it is sufficient to establish it in the case where the F_i and G are products of complete functions. When the F_k are of the form $F_k = S_{i_k}$ and $G = S^J = S_{j_1} S_{j_2} \cdots S_{j_s}$, the formula

$$(S_{i_1} S_{i_2} \cdots S_{i_r}) * S^J = \mu_r \left[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * \Delta^r S^J \right] \tag{73}$$

is equivalent to the multiplication formula (72). Next, let $I^{(k)} = (i_1^{(k)}, i_2^{(k)}, \ldots, i_{n_k}^{(k)})$ for every $k = 1, \ldots, r$, and consider the transformations

$$\mu_r \left[(S^{I^{(1)}} \otimes \cdots \otimes S^{I^{(r)}}) * \Delta^r G \right]$$

$$= \mu_r \left[\sum_{(G)} (S^{I^{(1)}} * G_{(1)}) \otimes (S^{I^{(2)}} * G_{(2)}) \otimes \cdots \otimes (S^{I^{(r)}} * G_{(r)}) \right]$$

(using Sweedler's notation for $\Delta^r G$)

$$= \mu_r \left[\sum_{(G)} \mu_{n_1}\Big((S_{i_1^{(1)}} \otimes \cdots \otimes S_{i_{n_1}^{(1)}}) * \Delta^{n_1} G_{(1)} \Big) \otimes \cdots \otimes \mu_{n_r}\Big((S_{i_1^{(r)}} \otimes \cdots \otimes S_{i_{n_r}^{(r)}}) * \Delta^{n_r} G_{(r)} \Big) \right]$$

(by application of formula (73))

$$= \mu_r \circ (\mu_{n_1} \otimes \cdots \otimes \mu_{n_r}) \left[\Big(S_{i_1^{(1)}} \otimes \cdots \otimes S_{i_{n_1}^{(1)}} \otimes S_{i_1^{(2)}} \otimes \cdots \otimes S_{i_{n_r}^{(r)}} \Big) * (\Delta^{n_1} \otimes \cdots \otimes \Delta^{n_r}) \circ \Delta^r G \right]$$

$$= \mu_n \left[\Big(S_{i_1^{(1)}} \otimes \cdots \otimes S_{i_{n_1}^{(1)}} \otimes \cdots \otimes S_{i_1^{(r)}} \otimes \cdots \otimes S_{i_{n_r}^{(r)}} \Big) * \Delta^N G \right]$$

(by associativity and coassociativity, with $N = n_1 + \cdots + n_r$)

$$= \Big(S^{I^{(1)}} S^{I^{(2)}} \cdots S^{I^{(r)}} \Big) * G \ .$$

□

Example 5.3 To compute $S^{212} * S^{23}$, one has to enumerate the matrices with row sums vector $I = (2, 1, 2)$ and column sums vector $J = (2, 3)$, which yields

$$S^{212} * S^{23} = 2\, S^{212} + S^{2111} + S^{1112} + S^{11111} \ .$$

Applying Proposition 5.2 and taking into account the fact that $S_k * F = F$ for $F \in \mathbf{Sym}_k$ and $S_k * F = 0$ otherwise, we obtain as well

$$\mu \left[(S^{21} \otimes S^2) * \Delta S^{23} \right] = \mu \left[(S^{21} \otimes S^2) * \left\{ \begin{array}{c} (S^2 \otimes 1 + S^1 \otimes S^1 + 1 \otimes S^2) \times \\ (S^3 \otimes 1 + S^2 \otimes S^1 + S^1 \otimes S^2 + 1 \otimes S^3) \end{array} \right\} \right]$$

$$= \mu \left[(S^{21} \otimes S^2) * (S^{21} \otimes S^2 + S^{12} \otimes S^{11} + S^3 \otimes S^2) \right]$$

$$= 2\, S^{212} + S^{1112} + S^{11111} + S^{2111} = S^{212} * S^{23} \ .$$

Example 5.4 One can recognize the involution $\eta : F \mapsto F * \lambda(1)$ by computing the products $S^I * \lambda(1)$. If $I = (i_1, \ldots, i_r)$,

$$S^I * \lambda(1) = \mu_r \left[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\lambda(1) \otimes \cdots \otimes \lambda(1)) \right]$$

$$= \Lambda_{i_1} \Lambda_{i_2} \cdots \Lambda_{i_r} = \Lambda^I \ .$$

Hence, η is the involutive automorphism whichs sends S_k to Λ_k. One can check that its action on ribbons is given by

$$\eta(R_I) = R_I * \lambda(1) = R_{\overline{I^\sim}} \ . \tag{74}$$

The involution $F \mapsto \lambda(1) * F$ is more easily identified in terms of permutations. Since the descent set of $\sigma\omega$ is the complement in $\{1, \ldots, n-1\}$ of the descent set of σ, one sees that it is the antiautomorphism

$$\lambda(1) * R_I = R_{I^\sim} = \omega(R_I) \ .$$

Other properties of the internal product can be lifted to the noncommutative case. For example, in [At], Atkinson defines maps $\varepsilon_I : \ \mathbf{Sym}_n \longrightarrow \mathbf{Sym}_{i_1} \otimes \mathbf{Sym}_{i_2} \otimes \cdots \otimes \mathbf{Sym}_{i_r}$ by setting

$$\varepsilon_I(F) = (S_{i_1} \otimes S_{i_2} \otimes \cdots \otimes S_{i_r}) * \Delta^r(F) \tag{75}$$

and shows that they are algebra homomorphisms for the internal product. He then uses them to construct a simplified version of the representation theory of the descent algebra (the first theory was due to Garsia and Reutenauer (*cf.* [GaR])). The fact that these maps are homomorphisms is equivalent to the following important property whose proof may be adapted from Atkinson's paper.

Proposition 5.5 *The iterated coproduct Δ^r is a homomorphism for the internal product from* **Sym** *into* $\mathbf{Sym}^{\otimes r}$. *In other words, for $F, G \in$* **Sym**

$$\Delta^r(F * G) = \Delta^r F * \Delta^r G \ .$$

In the commutative case, Sym_n endowed with the internal product is interpreted as the representation ring $R(\mathbf{S}_n)$ of the symmetric group, and $Sym_{i_1} \otimes Sym_{i_2} \otimes \cdots \otimes Sym_{i_r}$ as the representation ring of a Young subgroup $\mathbf{S}_I = \mathbf{S}_{i_1} \times \mathbf{S}_{i_2} \times \cdots \times \mathbf{S}_{i_r}$. With this interpretation, the maps ε_I correspond to the restriction of representations from $\mathbf{S}_n$ to $\mathbf{S}_I$.

The following consequence is worth noting. Remark that it follows from the structure theory of Hopf algebras and from Proposition 3.10 that **Sym** is the universal enveloping algebra of the free Lie algebra $L(\Psi)$ constituted by its primitive elements.

Corollary 5.6 *The internal product preserves the primitive Lie algebra $L(\Psi)$ of* **Sym**.

5.2 Lie idempotents

A Lie idempotent is an idempotent of the group algebra $K[\mathbf{S}_n]$ which acts as a projector from the free algebra $K\langle A\rangle$ onto the free Lie algebra $L(A)$ (for the right action of the symmetric group on words). It turns out that most of the Lie idempotents which have been encountered up to now are elements of the descent algebra (*cf.* [BBG] or [Re]). In this section, we show that these elements appear quite naturally in the theory of noncommutative symmetric functions.

In the commutative case, the products of power sums ψ^I are idempotent (up to a scalar factor) for the internal product. The noncommutative case is more complicated, but the simplest noncommutative internal products of power sums already lead to two interesting Lie idempotents.

Proposition 5.7 *For all $n \geq 1$, one has $\Psi_n * \Psi_n = n\,\Psi_n$.*

Proof — Recall that the generating series for the Ψ_n is given by

$$\psi(t) = \sum_{n\geq 1} t^{n-1}\,\Psi_n = \lambda(-t)\,\sigma'(t)$$

and that $\Delta\psi(t) = \psi(t)\otimes 1 + 1\otimes\psi(t)$. Then, using the fact that $\Psi_i * \Psi_j = 0$ for $i \neq j$ and applying Proposition 5.2, we can write

$$\sum_{n\geq 1} (xy)^{n-1}(\Psi_n * \Psi_n) = \psi(x) * \psi(y) = \mu\,[(\lambda(-x)\otimes\sigma'(x)) * (\psi(y)\otimes 1 + 1\otimes\psi(y))]$$

$$= \mu\,[(\lambda(-x) * 1)\otimes(\sigma'(x) * \psi(y))]$$

(since $\sigma'(x)$ has no term of weight 0)

$$= \left(\sum_{n\geq 1} nx^{n-1}\,S_n\right) * \left(\sum_{n\geq 1} y^{n-1}\,\Psi_n\right) = \sum_{n\geq 1} (xy)^{n-1}\,n\,\Psi_n\ ,$$

the last equality following from the fact that $S_n * F = F$ for $F \in \mathbf{Sym}_n$. □

This proposition is in fact equivalent to Dynkin's characterization of Lie polynomials (see *e.g.* [Re]). Indeed, recall that the *standard left bracketing* of a word $w = x_1x_2\cdots x_n$ in the noncommuting indeterminates x_i is the Lie polynomial

$$L_n(w) = [\cdots[[[x_1,x_2],x_3],x_4],\ldots,x_n]\ .$$

In terms of the right action of the symmetric group $\mathbf{S}_n$ on the homogeneous component $K_n\langle A\rangle$ of degree n of the free associative algebra $K\langle A\rangle$, defined on words by

$$x_1\,x_2\,\cdots\,x_n\cdot\sigma = x_{\sigma(1)}\,x_{\sigma(2)}\,\cdots\,x_{\sigma(n)}\ ,$$

one can write

$$L_n(w) = \sum_{\sigma\in\mathbf{S}_n} a_\sigma\,(w\cdot\sigma) = w\cdot\theta_n$$

where θ_n is an element of $\mathbf{Z}[\mathbf{S}_n]$. It is not difficult to show that θ_n is in fact in Σ_n and that one has (*cf.* [Ga])

$$\theta_n = \sum_{k=0}^{n-1} (-1)^k D_{=\{1,2,\ldots,k\}} .$$

To see this, one just has to write the permutations appearing in the first θ_i as ribbon tableaux and then to use induction. For example,

$$\theta_3 = [[1,2],3] = 123 \; - \; \begin{matrix} 2 & \\ 1 & 3 \end{matrix} \; - \; \begin{matrix} 3 & \\ 1 & 2 \end{matrix} \; + \; \begin{matrix} 3 \\ 2 \\ 1 \end{matrix}$$

and it is clear that when expanding $\theta_4 = [\theta_3, 4]$ one will only get those (signed) tableaux obtained from the previous ones by adding 4 at the end of the last row, minus those obtained by adding 4 on the top of the first column. Thus, in terms of noncommutative symmetric functions, we have from Corollary 3.14

$$\alpha(\theta_n) = \Psi_n ,$$

so that Proposition 5.7 becomes Dynkin's theorem : $\theta_n^2 = n\,\theta_n$, or more explicitely, a noncommutative polynomial $P \in K\langle X \rangle$ is a Lie polynomial iff $L_n(P) = nP$.

The same kind of argument works as well for the power sums of the second kind Φ_n.

Proposition 5.8 *For every $n \geq 1$, one has $\Phi_n * \Phi_n = n\,\Phi_n$.*

Proof — Using the generating series

$$\Phi(t) = \sum_{n\geq 1} \Phi_n \frac{t^n}{n} = \log \sigma(t) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} (tS_1 + t^2 S_2 + \cdots)^k ,$$

we have

$$\Phi(x) * \Phi(y) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (xS_1 + x^2 S_2 + \cdots)^n * \Phi(y) .$$

But, using Proposition 5.2, one has for $n > 1$

$$(xS_1 + x^2 S_2 + \cdots)^n * \Phi(y) = \mu_n \left[\left(\sum_{i_1 \geq 1} x^{i_1} S_{i_1} \right) \otimes \cdots \otimes \left(\sum_{i_n \geq 1} x^{i_n} S_{i_n} \right) * \Delta^n \Phi(y) \right] = 0 ,$$

since $\Phi(y)$ is primitive and $\sum_{i\geq 1} x^i S_i$ has no term of weight zero. Thus, using again the fact that S_n is a unit for the internal product on $\mathbf{Sym}_n$, we get

$$\Phi(x) * \Phi(y) = \left(\sum_{i\geq 1} x^i S_i \right) * \left(\sum_{i\geq 1} y^i \frac{\Phi_i}{i} \right) = \sum_{i\geq 1} (xy)^i \frac{\Phi_i}{i} = \Phi(xy) .$$

□

The element $e_n^{[1]} = \alpha^{-1}(\Phi_n/n)$ is thus an idempotent of the descent algebra Σ_n. In terms of permutations, the formula $\phi(t) = \log(1 + (tS_1 + t^2 S_2 + \cdots))$ shows that

$$e_n^{[1]} = \sum_{A \subseteq \{1,\ldots,n-1\}} \frac{(-1)^{|A|}}{|A|+1} D_{\subseteq A} .$$

(compare with [Re] p. 66-67). This idempotent is also a projector onto the free Lie algebra. One way to see this is to compute the products $\Phi_n * \Psi_n$ and $\Psi_n * \Phi_n$ and to use Dynkin's characterization of Lie polynomials.

Proposition 5.9 *For every $n \geq 1$, one has (i) $\Psi_n * \Phi_n = n\,\Phi_n$ and (ii) $\Phi_n * \Psi_n = n\,\Psi_n$.*

Proof — *(i)* Using Proposition 5.2 and the fact that $\Phi(y)$ is primitive for Δ, we have

$$\psi(x) * \phi(y) = (\lambda(-x)\,\sigma'(x)) * \Phi(y) = \mu\left[(\lambda(-x) \otimes \sigma'(x)) * (\Phi(y) \otimes 1 + 1 \otimes \Phi(y))\right]$$

$$= \sigma'(x) * \Phi(y) = y \sum_{n \geq 1} (xy)^{n-1} \Phi_n \ .$$

(ii) As above, one can write

$$\Phi(x) * \psi(y) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \left(\sum_{i \geq 1} x^i\, S_i \right)^n * \psi(y) = \left(\sum_{i \geq 1} x^i\, S_i \right) * \psi(y) = x\, \psi(xy) \ .$$

□

As already mentioned in Note 4.2, ribbon Schur functions appear in the commutative theory as the characteristics associated to a certain decomposition of the regular representation of $\mathbf{S}_n$. To give a precise statement, we need the notion (due to MacMahon, *cf.* [MM]) of *major index* of a permutation. Let $\sigma \in \mathbf{S}_n$ have descent set

$$\mathrm{Des}\,(\sigma) = \{d_1, \ldots, d_r\} \subseteq \{1, \ldots, n-1\} \ .$$

Then, by definition, $\mathrm{maj}\,(\sigma) = d_1 + \cdots + d_r$. We also define the major index of a composition I to be the major index of any permutation with descent composition I. That is, if $I = (i_1, i_2, \ldots, i_m)$, then

$$\mathrm{maj}\,(I) = (m-1)i_1 + (m-2)i_2 + \cdots + i_{m-1} \ .$$

Now, if $\mathcal{H}_n \subset \mathbf{C}[x_1, \ldots, x_n]$ denotes the vector space of $\mathbf{S}_n$-harmonic polynomials, and $\mathcal{H}_n^k$ its homogeneous component of degree k, it is a classical result that its graded characteristic as a $\mathbf{S}_n$-module is given by

$$\mathcal{F}_q(\mathcal{H}_n) := \sum_{k \geq 0} q^k\, \mathcal{F}(\mathcal{H}_n^k) = (q)_n\, S_n\left(\frac{X}{1-q}\right) \ , \tag{76}$$

where as usual $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$, the symmetric functions of $X/(1-q)$ being defined by the generating series

$$\sigma\left(\frac{X}{1-q}\,,\, t\right) = \prod_{k \geq 0} \sigma(X, tq^k) \ . \tag{77}$$

On the other hand, it is also known that

$$(q)_n\, S_n\left(\frac{X}{1-q}\right) = \sum_{|C|=n} q^{\mathbf{maj}\,(C)}\, R_C \ , \tag{78}$$

where the sum runs over all the compositions C of n, so that

$$\mathcal{F}(\mathcal{H}_n^k) = \sum_{\mathrm{maj}\,(C)=k} R_C \ . \tag{79}$$

It is therefore of interest to investigate the properties of the noncommutative symmetric functions $K_n(q)$ defined by

$$K_n(q) = \sum_{|C|=n} q^{\mathrm{maj}\,(C)}\, R_C \ . \tag{80}$$

These functions can be seen as noncommutative analogs of the Hall-Littlewood functions indexed by column shapes $Q_{(11\ldots 1)}(X/(1-q)\,;\,q)$ (*cf.* [McD]). One can describe their generating function

$$\mathbf{k}(q) = \sum_{n\geq 0} \frac{K_n(q)}{(q)_n}$$

in the following way.

Proposition 5.10 *One has*

$$\mathbf{k}(q) = \overleftarrow{\prod_{k\geq 0}}\ \sigma(q^k) = \cdots\ \sigma(q^3)\,\sigma(q^2)\,\sigma(q)\,\sigma(1)\ .$$

Proof — Expanding the product, we have

$$\overleftarrow{\prod_{k\geq 0}}\ \sigma(q^k) = \cdots(\cdots + q^{n_1 i_1} S_{i_1} + \cdots)\cdots(\cdots + q^{n_2 i_2} S_{i_2} + \cdots)\cdots(\cdots + q^{n_r i_r} S_{i_r} + \cdots)\cdots$$

$$= \sum_I \Big(\sum_{n_1>n_2>\ldots>n_r\geq 0} q^{n_1 i_1 + n_2 i_2 + \cdots + n_r i_r} \Big)\, S^I$$

$$= \sum_I q^{\mathrm{maj}\,(I)} \Big(\sum_{m_1\geq m_2\geq\ldots\geq m_r\geq 0} (q^{i_1})^{m_1}\,(q^{i_2})^{m_2}\ \cdots\ (q^{i_r})^{m_r} \Big)\, S^I$$

$$= \sum_I \frac{q^{\mathrm{maj}\,(I)}}{(1-q^{i_1})(1-q^{i_1+i_2})\cdots(1-q^{i_1+i_2+\cdots+i_r})}\, S^I\ .$$

Let $F_n(q)$ be the term of weight n in this series. We want to show that $(q)_n F_n(q) = K_n(q)$. Working with subsets of $[n-1] = \{1,\ldots,n-1\}$ rather than with compositions of n, we can write

$$(q)_n\, F_n(q) = \sum_{A\subseteq[n-1]} S^{c(A)}\, f(A)$$

where $f(A) = \prod_{t\in A} q^t \prod_{s\notin A}(1-q^s)$, so that

$$(q)_n\, F_n(q) = \sum_{A\subseteq[n-1]} f(A)\, \Big(\sum_{B\subseteq A} R_{c(B)} \Big) = \sum_{B\subseteq[n-1]} R_{c(B)}\, \Big(\sum_{A\supseteq B} f(A) \Big)$$

But, denoting by $\overline{A}$ the complementary subset of A in $[n-1]$ and by $\Sigma(A) = \mathrm{maj}\,(c(A))$ the sum of all elements of A, we see that

$$f(A) = q^{\Sigma(A)} \Big(\sum_{\overline{C}\subseteq\overline{A}} (-1)^{|\overline{C}|}\, q^{\Sigma(\overline{C})} \Big) = \sum_{\overline{C}\subseteq\overline{A}} (-1)^{|\overline{C}|}\, q^{\Sigma(A\cup\overline{C})}$$

$$= \sum_{B \supseteq A} (-1)^{|B|-|A|} q^{\Sigma(B)} .$$

It follows now by Möbius inversion in the lattice of subsets of $[n-1]$ that

$$\sum_{A \supseteq B} f(A) = q^{\Sigma(B)} = q^{\mathrm{maj}\,(c(B))} ,$$

as required. □

This factorization of $\mathbf{k}(q)$ shows in particular that $\mathbf{k}(q)$ is grouplike for Δ, which is well suited for computing internal products of the form $F * \mathbf{k}(q)$ by means of Proposition 5.2. For example, with $J = (j_1, \ldots, j_m)$, we have

$$S^J * \mathbf{k}(q) = \mu_m \left[(S_{j_1} \otimes \cdots \otimes S_{j_m}) * (\mathbf{k}(q) \otimes \cdots \otimes \mathbf{k}(q)) \right]$$

$$= \frac{K_{j_1}(q) \cdots K_{j_m}(q)}{(q)_{j_1} \cdots (q)_{j_m}} ,$$

and we obtain :

Proposition 5.11 *Let $J = (j_1, \ldots, j_m)$ be a composition of n. Then one has*

$$S^J * K_n(q) = \begin{bmatrix} n \\ j_1, j_2, \ldots, j_m \end{bmatrix}_q K_{j_1}(q)\, K_{j_2}(q) \cdots K_{j_m}(q) . \tag{81}$$

Denote by $\kappa_n(q)$ the element of $\mathbf{C}[q][\mathbf{S}_n]$ such that $\alpha(\kappa_n(q)) = \frac{1}{n} K_n(q) \in \mathbf{C}[q] \otimes \mathbf{Sym}$. Let $\zeta \in \mathbf{C}$ be a primitive nth root of unity. It has been shown by Klyachko (*cf.* [Kl]) that $\kappa_n(\zeta)$ is a Lie idempotent. To derive this result, we begin by considering the specialization at $q = \zeta$ of the multinomial coefficient in Proposition 5.11, which shows that $S^I * K_n(\zeta) = 0$ as soon as $\ell(I) > 1$. Hence, we get

$$\frac{\Phi_n}{n} * K_n(\zeta) = \log \sigma(1) * K_n(\zeta)$$

$$= \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \Big(\sum_{\ell(I)=r} S^I * K_n(\zeta) \Big) = \sum_{\ell(I)=1} S^I * K_n(\zeta) = K_n(\zeta) ,$$

from which it follows that the range of the associated endomophism $\kappa_n(\zeta)$ of $\mathbf{C}_n\langle A \rangle$ is contained in the free Lie algebra. To show that $\kappa_n(\zeta)$ is indeed an idempotent, we must prove that $\kappa_n(\zeta) * \Phi_n = \Phi_n$. As observed in [BBG], it is possible to get a little more.

Proposition 5.12 *For every $n \geq 1$, one has $K_n(q) * \Phi_n = (q)_{n-1}\, \Phi_n$.*

Proof — Let $\mathbf{k}_N(q) = \sigma(q^{N-1})\, \sigma(q^{N-2}) \cdots \sigma(1)$. According to Proposition 5.2, we have

$$\mathbf{k}_N(q) * \Phi_n = \mu_N \left[\left(\sigma(q^{N-1}) \otimes \sigma(q^{N-2}) \otimes \cdots \otimes \sigma(1) \right) * \Delta^N \Phi_n \right]$$

$$= \left(q^{n(N-1)} + q^{n(N-2)} + \cdots + q^n + 1 \right) \Phi_n$$

by primitivity of Φ_n, and taking now the limit for $N \to \infty$, we find $\mathbf{k}(q) * \Phi_n = (1-q^n)^{-1} \Phi_n$, whence $K_n(q) * \Phi_n = (q)_{n-1} \Phi_n$. □

Taking into account the fact that $(\zeta)_{n-1} = n$, we obtain Klyachko's result.

Corollary 5.13 *Let ζ be a primitive nth root of unity. Then, the element $\kappa_n(\zeta)$ of $\mathbf{C}[\mathbf{S}_n]$ defined by*

$$\kappa_n(\zeta) = \frac{1}{n}\sum_{\sigma\in\mathbf{S}_n} \zeta^{\mathrm{maj}\,(\sigma)}\,\sigma$$

is a Lie idempotent.

Note 5.14 The previous results show that it is natural to define noncommutative symmetric functions of the alphabet $\frac{1}{1-q}A$ by setting

$$S_n(\frac{1}{1-q}A) = \frac{K_n(q)}{(q)_n}\ .$$

It can be shown that

$$\lim_{q\to 1}\ (1-q^n)\,\Psi_n(\frac{1}{1-q}A) = \Phi_n(A)\ ,$$

which shows that the two families of noncommutative power sums are closely related.

Recall that the Witt numbers $\ell_n(k)$ are defined by

$$\ell_n(k) = \frac{1}{n}\sum_{d|n}\mu(d)\,k^{n/d}\ ,$$

μ being the usual Möbius function. We can then give the following simple characterization of Lie idempotents of the descent algebra in terms of noncommutative symmetric functions.

Theorem 5.15 *A symmetric function $J_n\in\mathbf{Sym}_n$ is the image under α of a Lie idempotent of Σ_n iff it has the form*

$$J_n = \frac{1}{n}\Psi_n\ +\ F_n$$

where $F_n\in L^2(\Psi) = [L(\Psi)\,,\,L(\Psi)]$. Thus, the Lie idempotents form an affine subspace in Σ_n. The dimension of this affine subspace is $d_1 = 1$ for $n=1$ and $d_n = \ell_n(2)-1$ for $n\geq 2$.

Proof — This will once more follow from Proposition 5.2. First, we see that $\Psi^I * \Psi_n = 0$ when $\ell(I)\geq 2$. Hence $F_n * \Psi_n = 0$ for every $F_n\in L^2(\Psi)\cap\mathbf{Sym}_n$. Since F_n is primitive, we also have

$$\begin{aligned}\Psi_n * F_n &= \mu\,[(\lambda(-1)\otimes\sigma'(1)) * (F_n\otimes 1 + 1\otimes F_n)]\\ &= \mu[0 + 1\otimes(\sigma'(1) * F_n)] = n\,F_n\end{aligned}$$

since $\sigma'(1)$ has no term of weight zero. So, for $F_n\in L^2(\Psi)\cap\mathbf{Sym}_n$ and $J_n = \frac{1}{n}\Psi_n + F_n$, one has $\Psi_n * J_n = n\,J_n$ and $J_n * \Psi_n = \Psi_n$, which shows that J_n is the image of a Lie idempotent.

Conversely, suppose that $\Psi_n * J_n = n\,J_n$ and $J_n * \Psi_n = \Psi_n$. Defining F_n by the relation $J_n = \frac{1}{n}\Psi_n + F_n$, we see that F_n must satisfy $F_n * \Psi_n = 0$, so that F_n must be of degree ≥ 2 in the generators Ψ_k, and that $\Psi_n * F_n = n\,F_n$, which implies that F_n is primitive. Indeed, Proposition 5.5 shows that

$$\Delta(n\,F_n) = \Delta\Psi_n * \Delta F_n = (\Psi_n\otimes 1 + 1\otimes\Psi_n) * (\sum_{(F)} F_{(1)}\otimes F_{(2)}\ )$$

$$= (\Psi_n * F_n) \otimes 1 + 1 \otimes (\Psi_n * F_n)$$

since any element $G \in \mathbf{Sym}$ satisfies $\Delta G = G \otimes 1 + 1 \otimes G + \sum_i P_i \otimes Q_i$, where P_i and Q_i are of weight ≥ 1.

The formula for the dimension follows from the Poincaré-Birkhoff-Witt theorem applied to the Lie algebra $L(\Psi)$ endowed with the weight graduation. If $h_n := \dim L_n(\Psi)$, where $L_n(\Psi)$ is the subspace of elements of weight n, then

$$\frac{1-t}{1-2t} = \prod_{n\geq 1} \left(\frac{1}{1-t^n}\right)^{h_n} .$$

Taking logarithms and applying Möbius inversion, one finds $h_n = \ell_n(2)$. □

Note 5.16 The fact that the dimension of the homogeneous component $L_n(\Psi)$ of *weight* n of the Lie algebra $L(\Psi)$ is $l_n(2)$ for $n \geq 2$ can be combinatorially interpreted. For $n \geq 2$, there is an explicit bijection between $L_n(\Psi)$ and the homogeneous component $L_n(a,b)$ of *degree* n of the free Lie algebra $L(a,b)$ on a two letters alphabet. This bijection follows from Lazard's elimination theorem (see [Bo]) which says in particular that the K-module $L(a,b)$ is the direct sum of two free Lie algebras

$$L(a,b) = K\, a \oplus L(\{(\mathrm{ad}\, a)^n \cdot b\ ,\ n \geq 0\})\ .$$

The desired bijection is obtained by considering the Lie morphism from $L(\Psi)$ into $L(a,b)$ mapping Ψ_n onto $(\mathrm{ad}\, a)^n \cdot b$ for $n \geq 1$.

Let us define a *quasi-idempotent* of a K-algebra $\mathcal{A}$ as an element π of $\mathcal{A}$ such that $\pi \cdot \pi = k\,\pi$ for some constant $k \in K$. Using this terminology, we can restate Theorem 5.15 in the following way.

Corollary 5.17 *Let π_n be an homogeneous element of $\mathbf{Sym}_n$. The following assertions are equivalent:*

1) *π_n is the image under α of a Lie quasi-idempotent of Σ_n.*
2) *π_n belongs to the Lie algebra $L(\Psi)$.*
3) *π_n is a primitive element for Δ.*

5.3 Eulerian idempotents

The generating series corresponding to other interesting families of idempotents also have a simple description in terms of noncommutative symmetric functions. Consider for example the coefficient of x^r in the expansion of $\sigma(1)^x$, that is, the series

$$E^{[r]} := \frac{1}{r!}\left(\sum_{k\geq 1} \frac{\Phi_k}{k}\right)^r = \sum_{n\geq r} E_n^{[r]} \tag{82}$$

where, using again the notation $\pi(I) = i_1\, i_2\, \cdots\, i_r$,

$$E_n^{[r]} = \frac{1}{r!} \sum_{|I|=n,\ell(I)=r} \frac{\Phi^I}{\pi(I)}\ .$$

It can be shown that the elements $e_n^{[r]} = \alpha^{-1}(E_n^{[r]})$ are idempotents of Σ_n, called *Eulerian idempotents*. They have been introduced independently by Mielnik and Plebański [MP] as ordering operators acting of series of iterated integrals and by Reutenauer (*cf.* [Re86]) as projectors associated with the canonical decomposition of the free associative algebra interpreted as the universal enveloping algebra of a free Lie algebra. They also appear in the work of Gerstenhaber and Schack (*cf.* [GS1]), where they are used to provide a Hodge-type decomposition for the Hochschild cohomology of a commutative algebra.

Here is a simple argument, due to Loday (*cf.* [Lod1] or [Lod2]), showing that the $e_n^{[r]}$ are indeed idempotents. Starting from the definition of $E^{[1]}$, *i.e.* $\sigma(1) = \exp E^{[1]}$, we have for any integer p

$$\sigma(1)^p = \exp(p\,E^{[1]}) = \sum_{k\geq 0} p^k \, \frac{(E^{[1]})^k}{k!} = \sum_{k\geq 0} p^k \, E^{[k]} \tag{83}$$

(setting $E^{[0]} = 1$). Now, denoting by $S_n^{[p]}$ the term of weight n in $\sigma(1)^p$, that is

$$S_n^{[p]} = \sum_{|I|=n,\ell(I)\leq p} S^I \ ,$$

we have

$$S_n^{[p]} = pE_n^{[1]} + p^2 E_n^{[2]} + \cdots + p^n E_n^{[n]} \tag{84}$$

so that the transition matrix from the $S_n^{[i]}$ to the $E_n^{[j]}$ is the Vandermonde matrix over $1, 2, \ldots, n$, and hence is invertible. Using the easily established fact that

$$S_n^{[p]} * S_n^{[q]} = S_n^{[pq]} \tag{85}$$

(see below), one deduces the existence of a decomposition $E_n^{[i]} * E_n^{[j]} = \sum_m a_{ijm}\, E_n^{[m]}$. Substituting this relation in (85) by means of (84), one obtains that the coefficients a_{ijm} must satisfy the equation

$$\sum_{1\leq i,j\leq n} p^i\, q^j\, a_{ijm} = (pq)^m$$

whose only solution is $a_{ijm} = 0$ except for $i = j = m$ in which case $a_{iii} = 1$. Note also that equation (84) with $p = 1$ shows that the sum of the $e_n^{[k]}$ is the identity of Σ_n, so that the $e_n^{[k]}$ form a complete family of othogonal idempotents.

Equation (85) is an instance of a quite general phenomenon. Indeed, in any Hopf algebra $\mathcal{A}$ with multiplication μ and comultiplication Δ, one can define *Adams operations* ψ^p by setting $\psi^p(x) = \mu_p \circ \Delta^p(x)$ (see [GS2] or [Lod3]). When we take for $\mathcal{A}$ the algebra **Sym** of noncommutative symmetric functions, these Adams operations are given by $\psi^p(F) = [\sigma(1)^p] * F$. Indeed, using Proposition 5.2, we have

$$[\sigma(1)^p] * F = \mu_p[(\sigma(1) \otimes \cdots \otimes \sigma(1)) * \Delta^p F] = \mu_p \circ \Delta^p(F) = \psi^p(F)$$

since $\sigma(1)$ is the unit element for $*$. This shows that $\sigma(1)^p * \sigma(1)^q = \sigma(1)^{pq}$, which is equation (85). In this particular case, the Adams operations can be interpolated into a one-parameter group : for any scalar $z \neq 0$, one can define $\psi^z(F) = \sigma(1)^z * F$.

¿From (85) we see that the $S_n^{[k]}$ generate a commutative $*$-subalgebra of $\mathbf{Sym}_n$ which is of dimension n according to (84). We shall call it the *Eulerian subalgebra* and denote it by $\mathbf{E}_n$. The corresponding subalgebra of Σ_n will also be called Eulerian, and will be denoted by $\mathcal{E}_n$.

Example 5.18 ¿From (84) we have $S_n^{[p]} * E_n^{[i]} = p^i E_n^{[i]}$, so that the minimal polynomial of $S_n^{[p]}$ is $f_n^{[p]}(x) := \prod_{1\le i\le n}(x - p^i)$. In [GS1], Gerstenhaber and Schack proved directly that the minimal polynomial of $s_n := \alpha^{-1}(S_n^{[2]} - S_n^{[1]})$ was $f_n^{[2]}(x-2)$, and then obtained the Eulerian idempotents by Lagrange interpolation, that is by setting

$$e_n^{[j]} = \prod_{i\neq j} (\lambda_i - \lambda_j)^{-1} (s_n - \lambda_i) \ ,$$

where $\lambda_i = 2^i - 2$. More precisely their idempotents are the images of those presented here by the automorphism of $\mathbf{Q}[\mathbf{S}_n]$ defined on permutations by $\sigma \mapsto \mathrm{sgn}\,(\sigma)\sigma$.

5.4 Eulerian symmetric functions

The commutative ribbon Schur functions are known to be related to the combinatorics of Eulerian and Euler numbers. These numbers count sets of permutations with constrained descents. To such a set, one can associate the sum of its elements in the descent algebra, and interpret it as a noncommutative symmetric function. In fact, most of the formulas satisfied by these numbers and their commutative symmetric analogs are specializations of identities at the level of noncommutative symmetric functions.

5.4.1 Noncommutative Eulerian polynomials

Let us recall that the Eulerian number $A(n,k)$ is defined as the number of permutations in S_n with exactly $k-1$ descents. Thus, $A(n,k)$ is equal to the number of standard ribbon-shaped Young tableaux, whose shape is encoded by a composition with exactly k parts, so that $A(n,k)/n!$ is the image by the specialization $S_i \longmapsto 1/i!$ of the symmetric function (introduced by Foulkes in [F2]) :

$$\mathbf{A}(n,k) = \sum_{\substack{|I|=n \\ \ell(I)=k}} R_I \ . \qquad (86)$$

These symmetric functions remain meaningful (and can in fact be better understood) in the noncommutative setting.

Definition 5.19 *Let t be an indeterminate commuting with the S_i. The noncommutative Eulerian polynomials are defined by*

$$\mathcal{A}_n(t) = \sum_{k=1}^{n} t^k \Big(\sum_{\substack{|I|=n \\ \ell(I)=k}} R_I \Big) = \sum_{k=1}^{n} \mathbf{A}(n,k)\, t^k \ . \qquad (87)$$

One has for these polynomials the following identity, whose commutative version is due to Désarménien (see [De]).

Proposition 5.20 *The generating series of the $\mathcal{A}_n(t)$ is*

$$\mathcal{A}(t) := \sum_{n\ge 0} \mathcal{A}_n(t) = (1-t)\ (1 - t\,\sigma(1-t))^{-1} \ . \qquad (88)$$

Proof — Using Proposition 4.13, it is not difficult to see that

$$\sum_{|I|=n} t^{\ell(I)} R_I = t \ (1 \ \ 1)^{\otimes(n-1)} \begin{pmatrix} 1 & 0 \\ -t & t \end{pmatrix}^{\otimes(n-1)} \begin{pmatrix} S_n \\ \vdots \\ S^{1^n} \end{pmatrix} = t \ (1-t \ \ t)^{\otimes(n-1)} \begin{pmatrix} S_n \\ \vdots \\ S^{1^n} \end{pmatrix} ,$$

which means that

$$\sum_{|I|=n} t^{\ell(I)} R_I = \sum_{|I|=n} (1-t)^{n-\ell(I)} \, t^{\ell(I)} \, S^I \ . \tag{89}$$

Hence,

$$\mathcal{A}(t) = \sum_I t^{\ell(I)} R_I = \sum_I (1-t)^{|I|-\ell(I)} \, t^{\ell(I)} \, S^I \ .$$

and it follows that

$$\mathcal{A}(t) = \sum_{i\geq 0} \left(\frac{t}{1-t}\right)^i \left(\sum_{j\geq 1} (1-t)^j \, S_j \right)^i = \sum_{i \geq 0} \left(\frac{t}{1-t}\right)^i (\, \sigma(1-t) - 1 \,)^i$$

$$= (\, 1 - \frac{t}{1-t} (\, \sigma(1-t) - 1 \,) \,)^{-1} \ .$$

□

Let us introduce the notation $\mathcal{A}_n^*(t) = (1-t)^{-n} \, \mathcal{A}_n(t)$. Using (89), we see that

$$\mathcal{A}^*(t) := \sum_{n\geq 0} \mathcal{A}_n^*(t) = \sum_I \left(\frac{t}{1-t}\right)^{\ell(I)} S^I \ .$$

This last formula can also be written in the form

$$\mathcal{A}^*(t) = \sum_{k \geq 0} \left(\frac{t}{1-t}\right)^k (S_1 + S_2 + S_3 + \cdots)^k \tag{90}$$

or

$$\frac{1}{1 - t\,\sigma(1)} = \sum_{n\geq 0} \frac{\mathcal{A}_n(t)}{(1-t)^{n+1}} \ . \tag{91}$$

We can also prove equation (90) directly as follows. Writing $t/(1-t) = x$, the right-hand side is

$$\sum_I S^I \, x^{\ell(I)} = \sum_I x^{\ell(I)} \ (\sum_{J \preceq I} R_J \ )$$

$$= \sum_J R_J \ (\sum_{J \preceq I} x^{\ell(I)} \ ) = \sum_J R_J \, x^{\ell(J)} \, (1+x)^{|J|-\ell(J)}$$

(using the bijection between compositions I of $n = |J|$ and subsets of $\{1, 2, \ldots, n-1\}$ and the fact that reverse refinement corresponds to inclusion)

$$= \sum_J R_J \, \frac{t^{\ell(J)}}{(1-t)^{\ell(J)}} \cdot \left(\frac{1}{1-t}\right)^{n-\ell(J)} = \sum_{n\geq 0} \mathcal{A}_n^*(t) \ .$$

It is not difficult to show, using these identities, that most classical formulas on Eulerian numbers (see *e.g.* [FS]) admit a noncommutative symmetric analog. Several of these formulas are already in the literature, stated in terms of permutations. The following ones have a rather simple interpretation in the noncommutative setting : they just give the transition matrices between several natural bases of the Eulerian subalgebras.

First, expanding the factors $(1-t)^{-(n+1)}$ in the right-hand side of (91) by the binomial theorem, and taking the coefficient of t^k in the term of weight n in both sides, we obtain

$$S_n^{[k]} = \sum_{i=0}^{k} \binom{n+i}{i} \mathbf{A}(n, k-i) \ , \tag{92}$$

an identity due to Loday, which shows in particular that the noncommutative Eulerian functions also span the Eulerian subalgebra. Similarly, one has

$$\frac{\mathcal{A}_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} t^k \, S_n^{[k]} \ ,$$

so that

$$\mathbf{A}(n,p) = \sum_{i=0}^{p} (-1)^i \binom{n+i}{i} S_n^{[p-i]} \ . \tag{93}$$

Another natural basis of the Eulerian subalgebra $\mathbf{E}_n$ is constituted by the elements

$$M_n^{[k]} := \sum_{|I|=n, \ell(I)=k} S^I \ . \tag{94}$$

Indeed, putting $x = t/(1-t)$, we have

$$\sum_{k \geq 0} x^k \, (S_1 + S_2 + \cdots)^k = \sum_{n \geq 0} \mathcal{A}_n^* \left(\frac{x}{1+x} \right) = \sum_{n \geq 0} (1+x)^n \, \mathcal{A}_n \left(\frac{x}{1+x} \right) \ ,$$

so that

$$\sum_{k=1}^{n} x^k \, M_n^{[k]} = \sum_{j=1}^{n} x^j \, (1+x)^{n-j} \, \mathbf{A}(n,j) \ . \tag{95}$$

Another kind of generating function can be obtained by writing

$$\sigma(1)^x = [1 + (S_1 + S_2 + \cdots)]^x = \sum_{k \geq 0} \binom{x}{k} M^{[k]} \ .$$

Comparing with $\sigma(1)^x = \exp(x \, E^{[1]})$, it follows that

$$\sum_{k=1}^{n} x^k \, E_n^{[k]} = \sum_{k=1}^{n} \binom{x}{k} M_n^{[k]} \ . \tag{96}$$

One obtains similarly the following expansion of the $E_n^{[k]}$ on the basis $\mathbf{A}(n,i)$, which is a noncommutative analog of Worpitzky's identity (see [Ga] or [Lod1]) :

$$\sum_{k=1}^{n} x^k \, E_n^{[k]} = \sum_{i=1}^{n} \binom{x-i+1}{n} \mathbf{A}(n,i) \ . \tag{97}$$

5.4.2 Noncommutative trigonometric functions

The Euler numbers (not to be confused with the Eulerian numbers of the preceding section), defined by

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x$$

can also be given a combinatorial interpretation (see [An]) which is again best understood in terms of symmetric functions (see [F1], [F2] or [De]). As shown by D. André, E_n is equal to the number of *alternating* permutations of S_n (σ is alternating if $\sigma_1 < \sigma_2 > \sigma_3 < \dots$). An alternating permutation is thus the same as a skew standard tableau of "staircase ribbon" shape, *i.e.* a ribbon shape indexed by the composition $C_{2k} = (2^k) = (2, 2, \dots, 2)$ if $n = 2k$ or $C_{2k+1} = (2^k 1) = (2, 2, \dots, 2, 1)$ if $n = 2k+1$. So the staircase ribbons provide symmetric analogs of the $E_n/n!$. As in the preceding section, it is possible to replace the commutative ribbons by the noncommutative ones. In fact, most of the important properties of the tangent numbers E_{2k+1} are specializations of general identities valid for the coefficients of the quotient of an odd power series by an even one. Such identities can also be given for noncommutative power series, but one has to distinguish between right and left quotients. The existence of a noncommutative interpretation of trigonometric functions has been first observed by Longtin (*cf.* [Lon], see also [MaR]).

Definition 5.21 *The noncommutative trigonometric functions associated to the noncommutative generic series $\sigma(t) = \sum_{n\geq 0} S_n t^n$ are*

$$SIN = \sum_{i\geq 0} (-1)^i S_{2i+1} , \quad COS = \sum_{i\geq 0} (-1)^i S_{2i}$$

$$SEC = (COS)^{-1} , \quad TAN_r = SIN \cdot (COS)^{-1} , \quad TAN_l = (COS)^{-1} \cdot SIN .$$

Definition 5.22 *The noncommutative Euler symmetric functions are defined by*

$$T_{2n} = R_{(2^n)} , \quad T_{2n+1}^{(r)} = R_{(1\,2^n)} , \quad T_{2n+1}^{(l)} = R_{(2^n\,1)} .$$

These symmetric functions give, as in the commutative case, the expansion of the secant and of the tangent :

Proposition 5.23 *One has the following identities :*

$$SEC = 1 + \sum_{n\geq 1} T_{2n} , \tag{98}$$

$$TAN_l = \sum_{n\geq 0} T_{2n+1}^{(l)} , \quad TAN_r = \sum_{n\geq 0} T_{2n+1}^{(r)} . \tag{99}$$

Proof — The three identities being proved in the same way, we only give the proof for the SEC function. In this case, it is sufficient to prove that

$$\left(\sum_{i\geq 0} (-1)^i R_{2^i} \right) \left(1 + \sum_{j\geq 1} R_{2^j} \right) = \sum_{i\geq 0} (-1)^i R_{2^i} + \sum_{i\geq 0, j\geq 1} (-1)^i R_{2^i} R_{2^j} = 1 . \tag{100}$$

But, according to Proposition 3.13,

$$R_{2^i}\, R_{2^j} = \begin{cases} R_{2^i,2^j} + R_{2^{i+2},2^{j-1}} & \text{when } j \geq 2 \\ R_{2^i,2} + R_{2^{i+2}} & \text{when } j = 1 \end{cases}$$

for $i \geq 1$. Hence the product in the left-hand side of relation (100) is equal to

$$\sum_{i\geq 0} (-1)^i\, R_{2^i} \;+ \sum_{j\geq 1} R_{2^j} + \sum_{i\geq 1} (-1)^i \,(\, R_{2^i,2} + R_{2^{i+2}}\,) + \sum_{i\geq 1, j\geq 2} (-1)^i \,(\, R_{2^i,2^j} + R_{2^{i+2},2^{j-1}}\,)\,,$$

which can be rewritten as follows

$$1 - R_2 + \sum_{j\geq 1} R_{2^j} + \sum_{i\geq 1} (-1)^i\, R_{2^i,2} + \sum_{i\geq 1, j\geq 2} (-1)^i\, R_{2^i,2^j} - \sum_{i\geq 2, j\geq 1} (-1)^i R_{2^i,2^j}$$

$$= 1 - R_2 + \sum_{j\geq 1} R_{2^j} + \sum_{i\geq 1, j\geq 1} (-1)^i\, R_{2^i,2^j} - \sum_{i\geq 2, j\geq 1} (-1)^i R_{2^i,2^j}$$

$$= 1 - R_2 + \sum_{j\geq 1} R_{2^j} - \sum_{j\geq 1} R_{2,2^j} = 1\ .$$

□

It can also be convenient to consider the symmetric hyperbolic functions

$$COSH = \sum_{i\geq 0} S_{2i}\,, \quad SINH = \sum_{i\geq 0} S_{2i+1}\,, \quad SECH = (COSH)^{-1}\,,$$

$$TANH_l = SINH \cdot SECH \quad \text{and} \quad TANH_r = SECH \cdot SINH\ .$$

Then, arguing as in the proof of Proposition 5.23, one can check by means of Proposition 3.13 that one has the following expansions :

$$SECH = 1 + \sum_{n\geq 1} (-1)^n\, T_{2n}\,,$$

$$TANH_l = \sum_{n\geq 0} (-1)^n\, T_{2n+1}^{(l)} \quad \text{and} \quad TANH_r = \sum_{n\geq 0} (-1)^n\, T_{2n+1}^{(r)}\ .$$

As an illustration of the use of these functions let us consider the sums of hooks

$$H_n = \sum_{k=0}^{n-1} R_{1^k,n-k}\ . \tag{101}$$

In the commutative case these functions appear for example as the moments of the symmetric Bessel polynomials (see [Le], [LT]). Here, we compute the inverse of their generating series :

Proposition 5.24 *Let* $\mathbf{H} = \sum_{n\geq 0} H_n$ *with* $H_0 = 1$*. Then, one has*

$$\mathbf{H}^{-1} = 1 - \sum_{n\geq 0} (-1)^n\, T_{2n+1}^{(r)} = 1 - TANH_r\ .$$

Proof — Since $1 - TANH_r = COSH^{-1}(COSH - SINH)$, the identity to be proved is equivalent to

$$\Big(\sum_{i\ge 0} (-1)^i R_i\Big)\Big(1 + \sum_{k\ge 0, l\ge 1} R_{1^k,l}\Big) = \sum_{j\ge 0} R_{2i} \ ,$$

which can be rewritten as

$$\sum_{i\ge 0} (-1)^i R_i + \sum_{k,i\ge 0, l\ge 1} (-1)^i R_i R_{1^k,l} = \sum_{j\ge 0} R_{2i} \ . \qquad (102)$$

But Proposition 3.13 shows that

$$R_i R_{1^k,l} = \begin{cases} R_{i,1^k,l} + R_{i+1,1^{k-1},l} & \text{when } k \ge 1 \\ R_{i,l} + R_{i+l} & \text{when } k = 0 \end{cases} \ .$$

Hence the left-hand side of equation (102) can be rewritten in the following way

$$\sum_{i\ge 0} (-1)^i R_i + \sum_{k\ge 0, l\ge 1} R_{1^k,l} + \sum_{i,l\ge 1} (-1)^i (R_{i,l} + R_{i+l}) + \sum_{i,k,l\ge 1} (-1)^i (R_{i,1^k,l} + R_{i+1,1^{k-1},l}) \ ,$$

which is itself equal to

$$\sum_{i\ge 0} (-1)^i R_i + \sum_{k\ge 0, l\ge 1} R_{1^k,l} + \sum_{i,l\ge 1} (-1)^i R_{i,l} + \sum_{i,l\ge 1} (-1)^i R_{i+l}$$
$$+ \sum_{i,k,l\ge 1} (-1)^i R_{i,1^k,l} + \sum_{i\ge 2, k\ge 0, l\ge 1} (-1)^{i-1} R_{i,1^k,l} \ .$$

Using now the fact that one has

$$\sum_{i,l\ge 1} (-1)^i R_{i+l} = - \sum_{k\ge 1} R_{2k}$$

and

$$\sum_{i,k,l\ge 1} (-1)^i R_{i,1^k,l} = \sum_{i\ge 2, k,l\ge 1} (-1)^i R_{i,1^k,l} - \sum_{k\ge 2, l\ge 1} R_{1^k,l} \ ,$$

it is easy to see that this last expression is equal to

$$1 - \sum_{i\ge 0} R_{2i+1} + \sum_{l\ge 1} R_l + \sum_{l\ge 1} R_{1,l} + \sum_{i,l\ge 1} (-1)^i R_{i,l} + \sum_{i\ge 2, l\ge 1} (-1)^{i-1} R_{i,l} = \sum_{i\ge 0} R_{2i} \ .$$

□

5.5 Continuous Baker-Campbell-Hausdorff formulas

In this section, we shall obtain the main results of [BMP] and [MP] by means of computations with the internal product and noncommutative Eulerian polynomials. To explain the method, we first treat the problem of the continuous BCH exponents (*cf.* Section 4.10).

For us, the problem is to express $\Phi(t)$ in terms of the Ψ^I. The starting point is to write

$$\Phi(t) = \Phi(1) * \sigma(t) \qquad (103)$$

using $\sigma(t)$ as a reproducing kernel, and expressing it in the form

$$\sigma(t) = 1 + \sum_{r\geq 1} \int_0^t dt_1 \cdots \int_0^{t_{r-1}} dt_r \ \psi(t_r) \cdots \psi(t_1) \ .$$

The problem is thus reduced to the computation of $\Phi(1) * \psi(t_r) \cdots \psi(t_1)$, which is given by the following lemma:

Lemma 5.25 *Let $F_1, \ldots, F_r$ be primitive for Δ. Then,*

$$\Phi(1) * (F_1 \cdots F_r) = \sum_{\sigma \in \mathbf{S}_r} \frac{(-1)^r}{r} \binom{r-1}{d(\sigma)}^{-1} F_{\sigma(1)} \cdots F_{\sigma(r)} \ .$$

That is, if $\phi_r := e_r^{[1]} = \alpha^{-1}(\Phi_r/r)$,

$$\Phi(1) * (F_1 \cdots F_r) = (F_1 \cdots F_r) \cdot \phi_r \ ,$$

the right action of a permutation $\sigma \in \mathbf{S}_r$ on $F_1 \cdots F_r$ being as usual $(F_1 \cdots F_r) \cdot \sigma = F_{\sigma(1)} \cdots F_{\sigma(r)}$.

Proof — Since $\Phi(1) = \log(1 + (\sigma(1) - 1))$,

$$\Phi(1) * (F_1 \cdots F_r) = \sum_{k \geq 1} \frac{(-1)^k}{k} (\sigma(1) - 1)^k * (F_1 \cdots F_r) \ ,$$

and by Proposition 5.2

$$(\sigma(1) - 1)^k * (F_1 \cdots F_r) = \mu_k \left[(\sigma(1) - 1)^{\otimes k} * \sum_{A_1, \ldots, A_k} F_{a_1^1} \cdots F_{a_{m_1}^1} \otimes \cdots \otimes F_{a_1^k} \cdots F_{a_{m_k}^k} \right]$$

where the sum runs over all decompositions $\{1, \ldots, r\} = A_1 \cup \cdots \cup A_k$ into disjoint subsets $A_i = \{a_1^i < \ldots < a_{m_i}^i\}$. This expression is equal to

$$\sum_{d(\sigma) \leq k-1} F_{\sigma(1)} \cdots F_{\sigma(r)} \ ,$$

so that

$$\Phi(1) * (F_1 \cdots F_r) = (F_1 \cdots F_r) \cdot \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{d(\sigma) \leq k-1} \sigma = (F_1 \cdots F_r) \cdot \phi_r \ .$$

$\square$

Using the step function

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

and the notations

$$\theta_{i,j} = \theta(t_i - t_j), \quad \Theta_n = \theta_{1,2} + \theta_{2,3} + \cdots + \theta_{n-1,n} \ ,$$

formula (71) can be rewritten in the form of [MP]

$$\Phi(t)=\sum_{r\geq 1}\int_0^t\cdots\int_0^t dt_1\cdots dt_r\frac{(-1)^{\Theta_r}}{r}\binom{r-1}{\Theta_r}^{-1}\psi(t_r)\cdots\psi(t_1) \tag{104}$$

which can be converted into a Lie series by means of Dynkin's theorem, since we know that $\Phi(t)$ is in the Lie algebra generated by the Ψ_i, that is,

$$\Phi(t)=\sum_{r\geq 1}\int_0^t\cdots\int_0^t dt_1\cdots dt_r\frac{(-1)^{\Theta_r}}{r^2}\binom{r-1}{\Theta_r}^{-1}\{\psi(t_r)\cdots\psi(t_1)\}\ , \tag{105}$$

where $\{\psi(t_r)\cdots\psi(t_1)\}=\mathrm{ad}\,\psi(t_r)\cdots\mathrm{ad}\,\psi(t_2)(\psi(t_1))$.

Looking at the proof of Lemma 5.25, one observes that the argument only depends of the fact that $\Phi(1)$ is of the form $g(\sigma(1)-1)$, where $g(t)=\sum_{n\geq 0}g_nt^n$ is a power series in one variable. More precisely, one has the following property, which allows for a similar computation of any series of the form $g(\sigma(t)-1)$.

Lemma 5.26 *Let $g(t)=\sum_{n\geq 0}g_nt^n\in K[[t]]$ be a formal power series in one variable, $G(t):=g(\sigma(t)-1)=\sum_{n\geq 0}G_nt^n$, $G_n\in\mathbf{Sym}_n$, and $\gamma_n:=\alpha(G_n)\in\Sigma_n$. Then, if the series $F_1,\ldots,F_r$ are primitive elements for Δ,*

$$G(1)*(F_1\cdots F_r)=(F_1\cdots F_r)\cdot\gamma_r\ .$$

□

Using $\sigma(t)$ as reproducing kernel as in (103), one obtains:

Corollary 5.27 *The expression of $G(t)$ in the basis Ψ^I is given by*

$$G(t)=\sum_{r\geq 0}\int_0^t dt_1\cdots\int_0^{t_{r-1}}dt_r(\psi(t_r)\cdots\psi(t_1))\cdot\gamma_r\ .$$

□

In particular, with $g(t)=(1+t)^x$, one finds

$$\sigma(t)^x=1+\sum_{r\geq 1}\int_0^t dt_1\cdots\int_0^{t_{r-1}}dt_r\sum_{k\geq 0}x^k\,(\psi(t_r)\cdots\psi(t_1))\cdot e_r^{[k]} \tag{106}$$

where the $e_r^{[k]}$ are the Eulerian idempotents. Using the expression of Eulerian idempotents on the basis of descent classes and changing the variables in the multiple integrals, one can as above get rid of the summations over permutations by introducing an appropriate kernel.

As explained in [BMP] and [MP], such a transformation is always possible, and to find the general expression of the kernel associated to an arbitrary analytic function $g(t)$, we just have to find it for the particular series

$$f_z(t)=\frac{1}{z-t}:=\sum_{n\geq 0}\frac{1}{z^{n+1}}t^n$$

since by the Cauchy integral formula

$$G(t) = \frac{1}{2\pi i}\oint_{z=0} \frac{g(z)}{z-(\sigma(t)-1)}dz \ .$$

Using Corollary 5.27, we have

$$F_z(t) = \sum_{r\geq 0}\int_0^t dt_1 \cdots \int_0^{t_{r-1}} dt_r(\psi(t_r)\cdots\psi(t_1))\cdot\gamma_r$$

where the $\gamma_r = \alpha(F_{z,r})$ are given by

$$\sum_{r\geq 0} F_{z,r} = \frac{1}{z-(\sigma(1)-1)} = \frac{1}{z+1}\cdot\frac{1}{1-(z+1)^{-1}\sigma(1)} = \sum_{n\geq 0}\frac{\mathcal{A}_n\left(\frac{1}{z+1}\right)}{[1-(z+1)^{-1}]^{n+1}}$$

$$= \sum_{n\geq 0}\left(\frac{z+1}{z}\right)^{n+1}\mathcal{A}_n\left(\frac{1}{z+1}\right) = \sum_I \frac{(z+1)^{|I|-(\ell(I)-1)}}{z^{|I|+1}} R_I$$

(by formula (91)). Thus,

$$\gamma_r = \frac{1}{z^{r+1}}\sum_{\sigma\in\mathbf{S}_r}(z+1)^{r-d(\sigma)}\sigma$$

and

$$F_z(t) = \sum_{r\geq 0}\int_0^t\cdots\int_0^t dt_1\cdots dt_r\ \frac{1}{z^{r+1}}(z+1)^{\Theta_r}\ \psi(t_r)\cdots\psi(t_1)$$

so that

$$G(t) = \frac{1}{2\pi i}\oint_{z=0} g(z)F_z(t)dz$$

$$= \sum_{r\geq 0}\int_0^t\cdots\int_0^t dt_1\cdots dt_r\ K_r[g](t_1,\ldots,t_r)\ \psi(t_r)\cdots\psi(t_1)\ ,$$

the kernels being given by

$$K_r[g](t_1,\ldots,t_r) = \frac{1}{2\pi i}\oint_{z=0}\frac{g(z)}{z^{r+1}}(z+1)^{\Theta_r}dz \ . \tag{107}$$

6 Duality

As recalled in section 2.1, an important feature of the algebra Sym of commutative symmetric functions is that it is a self-dual Hopf algebra (for the coproduct Δ and the standard scalar product). Moreover, the commutative internal product $*$ is dual to the second coproduct $\delta : F \mapsto F(XY)$, and the two bialgebra structures are intimately related (*e.g.* each coproduct is a morphism for the product dual to the other one).

Such a situation cannot be expected for the noncommutative Hopf algebra **Sym**. A reason for this is that Δ is cocommutative, and cannot be dual to a noncommutative multiplication. One has thus to look for the dual bialgebra $\mathbf{Sym}^*$ of **Sym**. It follows from a recent work by Malvenuto and Reutenauer (*cf.* [MvR]) that this dual can be identified with another interesting generalisation of symmetric functions, that is, the algebra $\widehat{Qsym}$ of *quasi-symmetric functions*, whose definition is due to Gessel (see [Ge]).

6.1 Quasi-symmetric functions

Let $X = \{x_1 < x_2 < x_3 < \ldots\}$ be an infinite totally ordered set of *commuting* indeterminates. A formal series $f \in \mathbf{Q}[[X]]$ is said to be *quasi-symmetric* if for any two finite sequences $y_1 < y_2 < \ldots < y_k$ and $z_1 < z_2 < \ldots < z_k$ of elements of X, and any exponents $i_1, i_2, \ldots, i_k \in \mathbf{N}$, the monomials $y_1^{i_1}\, y_2^{i_2}\, \cdots\, y_k^{i_k}$ and $z_1^{i_1}\, z_2^{i_2}\, \cdots\, z_k^{i_k}$ have the same coefficient in f.

The quasi-symmetric series (resp. polynomials) form a subring denoted by $\widehat{Qsym}$ (resp. $Qsym$) of $\mathbf{Q}[[X]]$, naturally graded by the graduation $(\widehat{Qsym}_n)$ (resp. $(Qsym_n)$) inherited from $\mathbf{Q}[[X]]$. A natural basis of $Qsym_n$ is then provided by the *quasi-monomial functions*, defined by

$$M_I = \sum_{y_1 < y_2 < \ldots < y_k} y_1^{i_1}\, y_2^{i_2}\, \cdots\, y_k^{i_k}$$

where $I = (i_1, \ldots, i_k)$ is any composition of n. In particular, the dimension of $Qsym_n$ is $2^{n-1} = \dim \mathbf{Sym}_n$. Another convenient basis, also introduced in [Ge] is constituted by the functions

$$F_I = \sum_{J \succeq I} M_J$$

which we propose to call *quasi-ribbons.*

Let Y be a second infinite totally ordered set of indeterminates and denote by $X \hat{+} Y$ the ordered sum of X and Y, *i.e.* their disjoint union ordered by $x < y$ for $x \in X$ and $y \in Y$, and by the previous orderings on X and Y. Using the standard identification $\widehat{Qsym} \otimes \widehat{Qsym} \equiv \widehat{Qsym}(X, Y)$ (series which are separately quasi-symmetric in X and Y) defined by $f \otimes g \equiv f(X)\, g(Y)$, Malvenuto and Reutenauer (*cf.* [MvR]) defined a coproduct γ on $\widehat{Qsym}$ by setting

$$\gamma(f) = f(X \hat{+} Y) \tag{108}$$

for $f \in \widehat{Qsym}$. They show that $\widehat{Qsym}$ becomes then a Hopf algebra, with antipode ν given by

$$\nu(F_I) = (-1)^{|I|}\, F_{I^\sim} \tag{109}$$

where $C^\sim$ is the conjugate composition. These operations, when restricted to $Sym \subset \widehat{Qsym}$, coincide with the usual ones. That is, when f is a symmetric function, $\gamma(f) = f(X+Y)$ and $\nu(f) = f(-X)$.

As shown in [Ge], the other coproduct δ of Sym can also be extended to $\widehat{Qsym}$. Define $X\hat{\times}Y$ to be XY endowed with the lexicographic ordering, *i.e* $x_1y_1 < x_2y_2$ iff $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. The extension of δ to $\widehat{Qsym}$ is then defined by setting

$$\delta(f) = f(X\hat{\times}Y)$$

for $f \in \widehat{Qsym}$.

6.2 The pairing between Sym and $\widehat{Qsym}$

It has been shown in [Ge] that the dual space $Qsym_n^*$ endowed with the product adjoint to δ is anti-isomorphic to the descent algebra Σ_n. The dual Hopf algebra $\widehat{Qsym}^*$ of the whole algebra of quasi-symmetric series has been recently identified in [MvR]. If one introduces a pairing between $\widehat{Qsym}$ and **Sym** by setting

$$\langle M_I\, ,\, S^J\rangle = \delta_{IJ} \tag{110}$$

for every compositions I, J, the results of [Ge] and [MvR] which are relevant for our purposes can be summarized in the following theorem.

Theorem 6.1 *1. The pairing* (110) *induces an isomorphism of Hopf algebras, given by* $(S^J)^* \mapsto M_I$, *between the dual* **Sym*** *of* **Sym** *and the Hopf algebra* $\widehat{Qsym}$ *of quasi-symmetric series (or equivalently, an isomorphism between the graded dual* **Sym**^{*gr} *and the polynomial quasi-symmetric functions Qsym). More precisely, one has for* $f, g \in \widehat{Qsym}$ *and* $P, Q \in$ **Sym**

$$\langle f\, ,\, PQ\rangle = \langle \gamma f\, ,\, P\otimes Q\rangle \tag{111}$$

$$\langle fg\, ,\, P\rangle = \langle f\otimes g\, ,\, \Delta P\rangle \tag{112}$$

$$\langle f\, ,\, \tilde{\omega} P\rangle = \langle \nu f\, ,\, P\rangle\ . \tag{113}$$

2. Moreover, the coproduct δ *of* $\widehat{Qsym}$ *is dual to the internal product* $*$ *of* **Sym**:

$$\langle \delta f\, ,\, P\otimes Q\rangle = \langle f\, ,\, P * Q\rangle\ . \tag{114}$$

3. The quasi-ribbons are dual to the ribbons, i.e. $\langle F_I\, ,\, R_J\rangle = \delta_{IJ}$.

4. The antipode ν *of* $\widehat{Qsym}$ *is given by* $\nu F_C = (-1)^{|C|}\, F_{C^\sim}$.

5. Let τ *be any permutation with descent composition* $D(\tau) := c(\mathrm{Des}\,\tau) = C$. *Then,*

$$\delta F_C = \sum_{\sigma\pi=\tau} F_{D(\pi)} \otimes F_{D(\sigma)}\ . \tag{115}$$

6. If $g \in \widehat{Qsym}$ *is a symmetric function, then*

$$\langle g\, ,\, R_C\rangle = (g\, ,\, R_C) \tag{116}$$

where, in the right-hand side, R_C *stands for the* commutative *ribbon Schur function. In other words,* $g = \sum_C (g\, ,\, R_C)\, F_C$.

The pairing (110) can be better understood by means of an analog of the classical Cauchy formula of the commutative theory. Let A be a virtual noncommutative alphabet, and X a totally ordered commutative alphabet as above. Then, one can define the symmetric functions of the noncommutative alphabet XA by means of the generating series

$$\sigma(XA,1) = \overrightarrow{\prod_{x\in X}} \sigma(A,x) \ , \tag{117}$$

the above product being taken with respect to the total order of X. The expansion of this infinite product leads at once to the relation

$$\sigma(XA,1) = \sum_I M_I(X)\, S^I(A) \ . \tag{118}$$

Expanding each S^I on the ribbon functions, one obtains the following identities :

$$\sigma(XA,1) = \sum_I M_I(X)\,(\sum_{J\preceq I} R_J(A)\,) = \sum_J\,(\sum_{I\succeq J} M_I(X)\,)\, R_J(A) = \sum_J F_J(X)\, R_J(A)\ . \tag{119}$$

More generally, for any basis (U_I) of **Sym** with dual basis (V_I) in $Qsym$, one has

$$\sigma(XA,1) = \sum_I V_I(X)\, U_I(A) \ . \tag{120}$$

This property can be used to describe the dual bases of the various bases of **Sym**. Also, the proof of Proposition 5.10 can be interpreted as a computation of the specialization $X = \{1, q, q^2, \ldots\}$ of the quasi-symmetric functions M_I and F_I.

7 Specializations

In this section, we study several interesting cases of specialization of the symmetric functions defined in Section 3. In particular, we exhibit two realizations of the specialization $\Lambda_k = 0$ for $k > n$ by functions of n noncommuting variables which are symmetric in an appropriate sense. We also consider extensions of the theory to skew polynomial rings, and another kind of specialization associated with a noncommutative matrix. The use of these matrix symmetric functions is illustrated by some computations in the universal enveloping algebra $U(gl(n, \mathbf{C}))$.

In some cases, it is of interest to consider specializations of general quasi-Schur functions. On these occasions, the word specialization means 'specialization of the free field $K \not< S \not> := K \not< S_0, \mathbf{S}_1, S_2, \ldots \not>$', that is, a ring homomorphism η defined on a subring R_η of $K \not< S \not>$ containing **Sym** such that any element of R_η not in the kernel of η has an inverse in R_η. For more details on the category of fields and specializations, see [Co].

7.1 Rational symmetric functions of n noncommutative variables

We fix n noncommutative indeterminates $x_1, \ x_2, \ \ldots, \ x_n$, the variable t still being a commutative indeterminate. We set $x = t^{-1}$. In the commutative case, the quasi-determinant

$$g(x) = \begin{vmatrix} 1 & \ldots & 1 & 1 \\ x_1 & \ldots & x_n & x \\ x_1^2 & \ldots & x_n^2 & x^2 \\ \vdots & & \vdots & \vdots \\ x_1^n & \ldots & x_n^n & \boxed{x^n} \end{vmatrix}$$

reduces to the polynomial

$$(x - x_1)\,(x - x_2)\,\ldots\,(x - x_n) = t^{-n}\,(1 - t\,x_1)\,(1 - t\,x_2)\,\ldots\,(1 - t\,x_n)\ .$$

In the noncommutative case, basic properties of quasi-determinants imply that $g(x)$ is again a monic (left) polynomial of degree n, which vanishes under the substitution $x = x_i, \ i = 1, \ldots, n$. In fact, according to a theorem of Bray and Whaples [BW], if the x_i are specialized to n pairwise nonconjugate elements c_i of a division ring, then the polynomial

$$g(x) = \begin{vmatrix} 1 & \ldots & 1 & 1 \\ c_1 & \ldots & c_n & x \\ c_1^2 & \ldots & c_n^2 & x^2 \\ \vdots & & \vdots & \vdots \\ c_1^n & \ldots & c_n^n & \boxed{x^n} \end{vmatrix}$$

is the only monic polynomial of degree n such that $g(c_i) = 0, \ i = 1, \ldots, n$. Moreover, $g(x)$ has no other (right) roots, and any polynomial $h(x)$ having all the c_i as roots is right divisible by $g(x)$, that is, $h(x) = q(x)\,g(x)$ for some polynomial q.

Thus we are led in the noncommutative case to the following definition.

Definition 7.1 *The elementary symmetric functions $\Lambda_k(x_1,\ldots,x_n)$ are defined by*

$$\sum_{k\geq 0} \Lambda_k(x_1,\ldots,x_n)\,(-t)^k = \begin{vmatrix} 1 & \ldots & 1 & t^n \\ x_1 & \ldots & x_n & t^{n-1} \\ x_1^2 & \ldots & x_n^2 & t^{n-2} \\ \vdots & & \vdots & \vdots \\ x_1^n & \ldots & x_n^n & \boxed{1} \end{vmatrix} .$$

In other words we specialize the symmetric functions of Section 3 by setting

$$\lambda(-t) = \begin{vmatrix} 1 & \ldots & 1 & t^n \\ x_1 & \ldots & x_n & t^{n-1} \\ x_1^2 & \ldots & x_n^2 & t^{n-2} \\ \vdots & & \vdots & \vdots \\ x_1^n & \ldots & x_n^n & \boxed{1} \end{vmatrix} . \tag{121}$$

The expansion of the quasi-determinant $\lambda(-t)$ by its last row gives the following result.

Proposition 7.2 *For $0 \leq k \leq n$, we have*

$$\Lambda_k(x_1,\ldots,x_n) = (-1)^{k-1} \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ x_1^{n-k-1} & \ldots & x_n^{n-k-1} \\ x_1^{n-k+1} & \ldots & x_n^{n-k+1} \\ \vdots & & \vdots \\ x_1^{n-1} & \ldots & \boxed{x_n^{n-1}} \end{vmatrix} \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ x_1^{n-k} & \ldots & \boxed{x_n^{n-k}} \\ \vdots & & \vdots \\ x_1^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}^{-1} .$$

and $\Lambda_k(x_1,\ldots,x_n) = 0$ for $k > n$.

Example 7.3 For $n = 2$, we obtain

$$\Lambda_1(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ x_1^2 & \boxed{x_2^2} \end{vmatrix} \begin{vmatrix} 1 & 1 \\ x_1 & \boxed{x_2} \end{vmatrix}^{-1} = (x_2^2 - x_1^2)(x_2 - x_1)^{-1} , \tag{122}$$

$$\Lambda_2(x_1,x_2) = -\begin{vmatrix} x_1 & x_2 \\ x_1^2 & \boxed{x_2^2} \end{vmatrix} \begin{vmatrix} 1 & \boxed{1} \\ x_1 & x_2 \end{vmatrix}^{-1} = (x_2^2 - x_1x_2)(x_1^{-1}x_2 - 1)^{-1} . \tag{123}$$

It is not immediately clear on the expression (123) that $\Lambda_2(x_2,x_1) = \Lambda_2(x_1,x_2)$. However, one has the following proposition.

Proposition 7.4 *The $\Lambda_k(x_1,\ldots,x_n)$ are symmetric functions of the noncommutative variables x_1, x_2, $\ldots$, x_n, that is, they are invariant under any permutation of the x_k.*

Proof — By Proposition 2.7, a quasi-determinant is invariant by any permutation of its columns. Thus, the generating series (121) is a symmetric function of x_1, $\ldots$, x_n. Hence, its coefficients are symmetric. □

We shall now compute the complete symmetric functions $S_k(x_1,\ldots,x_n)$.

Proposition 7.5 *For every $k, n \geq 0$, we have*

$$S_k(x_1,\ldots,x_n) = \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ x_1^{n-2} & \ldots & x_n^{n-2} \\ x_1^{n-k+1} & \ldots & \boxed{x_n^{n-k+1}} \end{vmatrix} \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \ldots & \boxed{x_n^{n-1}} \end{vmatrix}^{-1} . \tag{124}$$

Proof — Denote temporarily the right-hand side of (124) by $\overline{S}_k(x_1,\ldots,x_n)$. To prove that $S_k(x_1,\ldots,x_n) = \overline{S}_k(x_1,\ldots,x_n)$, it is sufficient to check the identity

$$\begin{vmatrix} \overline{S}_1(x_1,\ldots,x_n) & \overline{S}_2(x_1,\ldots,x_n) & \ldots & \overline{S}_{n-1}(x_1,\ldots,x_n) & \boxed{\overline{S}_n(x_1,\ldots,x_n)} \\ \overline{S}_0(x_1,\ldots,x_n) & \overline{S}_1(x_1,\ldots,x_n) & \ldots & \overline{S}_{n-2}(x_1,\ldots,x_n) & \overline{S}_{n-1}(x_1,\ldots,x_n) \\ 0 & \overline{S}_0(x_1,\ldots,x_n) & \ldots & \overline{S}_{n-3}(x_1,\ldots,x_n) & \overline{S}_{n-2}(x_1,\ldots,x_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \overline{S}_0(x_1,\ldots,x_n) & \overline{S}_1(x_1,\ldots,x_n) \end{vmatrix}$$

$$= (-1)^{n-1}\Lambda_n(x_1,\ldots,x_n) .$$

This will follow from a slightly generalized form of Bazin's theorem for quasi-determinants. The computation is illustrated on an example. Take $n = k = 3$ and denote for short by $|i_1 i_2 \boxed{i_3}|$ the quasi-minor

$$|i_1 i_2 \boxed{i_3}| = \begin{vmatrix} x_1^{i_1} & x_2^{i_1} & x_3^{i_1} \\ x_1^{i_2} & x_2^{i_2} & x_3^{i_2} \\ x_1^{i_3} & x_2^{i_3} & \boxed{x_3^{i_3}} \end{vmatrix} .$$

Then one has

$$\begin{vmatrix} \overline{S}_1(x_1,x_2,x_3) & \overline{S}_2(x_1,\ldots,x_n) & \boxed{\overline{S}_3(x_1,x_2,x_3)} \\ \overline{S}_0(x_1,x_2,x_3) & \overline{S}_1(x_1,x_2,x_3) & \overline{S}_2(x_1,x_2,x_3) \\ 0 & \overline{S}_0(x_1,x_2,x_3) & \overline{S}_1(x_1,x_2,x_3) \end{vmatrix} = \begin{vmatrix} |01\boxed{3}| & |01\boxed{4}| & \boxed{|01\boxed{5}|} \\ |01\boxed{2}| & |01\boxed{3}| & |01\boxed{4}| \\ 0 & |01\boxed{2}| & |01\boxed{3}| \end{vmatrix} |01\boxed{2}|^{-1}$$

$$= \begin{vmatrix} |23\boxed{5}| & |12\boxed{5}| & \boxed{|01\boxed{5}|} \\ |23\boxed{4}| & |12\boxed{4}| & |01\boxed{4}| \\ |23\boxed{3}| & |12\boxed{3}| & |01\boxed{3}| \end{vmatrix} |01\boxed{2}|^{-1} = |34\boxed{5}||\boxed{2}34|^{-1}|01\boxed{2}||01\boxed{2}|^{-1} ,$$

$$= |12\boxed{3}||\boxed{0}12|^{-1} = \Lambda_3(x_1,x_2,x_3) .$$

Here, the second equality is obtained by multiplying the columns of the quasi-minors from the right by suitable powers of x_1, x_2, x_3. The third equality follows from Theorem 2.20. □

More generally, one can express a quasi-Schur function $\check{S}_I(x_1,\ldots,x_n)$ as the ratio of two quasi-minors of the Vandermonde matrix. This may be seen as a noncommutative analog of the classical expression of a Schur function as a ratio of two alternants.

Proposition 7.6 (Cauchy-Jacobi formula for quasi-Schur functions)
Let $I = (i_1,\ldots,i_m)$ be a partition of length $m \leq n$. Set

$$(s_1,\ldots,s_n) = (0,1,\ldots,n-1) + (0,\ldots,0,i_1,\ldots,i_m) ,$$

$$(t_1,\ldots,t_n) = (0,1,\ldots,n-1) + (0,\ldots,0,i_1-1,\ldots,i_{m-1}-1) \ .$$

Then,

$$\check{S}_I(x_1,\ldots,x_n) = (-1)^{m-1} \begin{vmatrix} x_1^{s_1} & \ldots & x_n^{s_1} \\ \vdots & \ddots & \vdots \\ x_1^{s_n} & \ldots & \boxed{x_n^{s_n}} \end{vmatrix} \begin{vmatrix} x_1^{t_1} & \ldots & x_n^{t_1} \\ \vdots & & \vdots \\ x_1^{t_{n-m+1}} & \ldots & \boxed{x_n^{t_{n-m+1}}} \\ \vdots & & \vdots \\ x_1^{t_n} & \ldots & x_n^{t_n} \end{vmatrix}^{-1} .$$

Proof — The proof is similar to the proof of Proposition 7.5. □

Of course all the formulas of Section 3 may be applied to the symmetric functions of $x_1,\ldots,x_n$ defined in this section. This is worth noting since a direct computation may be quite difficult even in the simplest cases. For example it is a good exercise to check by hand the formula

$$(x_2^2-x_1^2)(x_2-x_1)^{-1}(x_2^2-x_1^2)(x_2-x_1)^{-1} = (x_2^3-x_1^3)(x_2-x_1)^{-1} + (x_2^2-x_1x_2)(x_1^{-1}x_2-1)^{-1} \ ,$$

that is, $S_1(x_1,x_2)^2 = S_2(x_1,x_2) + \Lambda_2(x_1,x_2)$.

7.2 Rational (s,d)-symmetric functions of n noncommutative variables

We present here a natural extension of the case studied in Section 7.1. We fix in the same way n noncommutative variables $x_1,\ldots,x_n$. Let K be the skew field generated by these variables, s an automorphism of K and d a s-derivation of K. Consider an indeterminate X such that

$$X\,k = s(k)\,X + d(k)$$

for every $k \in K$. The polynomial algebra obtained in this way is denoted $K[X,s,d]$. Lam and Leroy have defined in this framework a notion of Vandermonde matrix associated with the family (x_i) which allows to extend the results of Section 7.1 to this more general context. Let us first give the following definition.

Definition 7.7 ([LL]) *Let k be an element of K. Its (s,d)-power $P_n(k)$ of order n is then inductively defined by*

$$P_0(k) = 1 \quad \textit{and} \quad P_{n+1}(k) = s(P_n(k))\,k + d(P_n(k)) \ .$$

This definition is motivated by the following result. Let $f(X) = \sum_i f_i X^i$ be a polynomial of $K[X,s,d]$. Then there exists a unique polynomial $q(t)$ such that

$$f(X) = q(X)\,(X-a) + \sum_i f_i\,P_i(a) \ .$$

This shows in particular that $\sum_i f_i\,P_i(a)$ is the good "evaluation" of $f(X)$ for $X = a$ and hence that $P_n(a)$ really plays the role in the (s,d)-context of the n-th power of a.

We can now introduce the elementary (s,d)-symmetric functions (defined in [LL]) which are given by the following specialization of the series $\lambda(t)$.

Definition 7.8 *The elementary symmetric functions $\Lambda_k^{(s,d)}(x_1,\ldots,x_n)$ are defined by*

$$\sum_{k\geq 0} \Lambda_k^{(s,d)}(x_1,\ldots,x_n)\,(-t)^k = \begin{vmatrix} 1 & \ldots & 1 & t^n \\ x_1 & \ldots & x_n & t^{n-1} \\ P_2(x_1) & \ldots & P_2(x_n) & t^{n-2} \\ \vdots & & \vdots & \vdots \\ P_n(x_1) & \ldots & P_n(x_n) & \boxed{1} \end{vmatrix} .$$

Expanding the quasi-determinant by its last column, we get the following result.

Proposition 7.9 *For every $k \in [0,n]$, we have*

$$\Lambda_k^{(s,d)}(x_1,\ldots,x_n) = (-1)^{k-1} \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ P_{n-k-1}(x_1) & \ldots & P_{n-k-1}(x_n) \\ P_{n-k+1}(x_1) & \ldots & P_{n-k+1}(x_n) \\ \vdots & & \vdots \\ P_n(x_1) & \ldots & \boxed{P_n(x_n)} \end{vmatrix} \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ P_{n-k}(x_1) & \ldots & \boxed{P_{n-k}(x_n)} \\ \vdots & & \vdots \\ P_{n-1}(x_1) & \ldots & P_{n-1}(x_n) \end{vmatrix}^{-1}$$

and $\Lambda_k^{(s,d)}(x_1,\ldots,x_n) = 0$ for every $k > n$.

Example 7.10 For $n = 2$, we obtain the following elementary symmetric functions

$$\Lambda_1^{(s,d)}(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ s(x_1)\,x_1 + d(x_1) & \boxed{s(x_2)\,x_2 + d(x_2)} \end{vmatrix} \begin{vmatrix} 1 & 1 \\ x_1 & \boxed{x_2} \end{vmatrix}^{-1}$$

$$= \left(s(x_2)\,x_2 - s(x_1)\,x_1 + d(x_2 - x_1)\right)(x_2 - x_1)^{-1} \ ,$$

$$\Lambda_2^{(s,d)}(x_1,x_2) = \begin{vmatrix} x_1 & x_2 \\ s(x_1)\,x_1 + d(x_1) & \boxed{s(x_2)\,x_2 + d(x_2)} \end{vmatrix} \begin{vmatrix} 1 & \boxed{1} \\ x_1 & x_2 \end{vmatrix}^{-1}$$

$$= \left(s(x_2)\,x_2 + d(x_2) - s(x_1)\,x_2 - d(x_1)\,x_1^{-1}\,x_2\right)(1 - x_1^{-1}\,x_2)^{-1} \ .$$

Again, Proposition 2.7 shows immediately that the $\Lambda_k^{(s,d)}(x_1,\ldots,x_n)$ are invariant under any permutation of the noncommutative variables x_i.

Using the same method as in Section 7.1, it is also possible to derive the following quasi-determinantal expression of the complete (s,d)-symmetric functions.

Proposition 7.11 *For every $k \geq 0$, we have*

$$S_k^{(s,d)}(x_1,\ldots,x_n) = \begin{vmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ \vdots & & \vdots \\ P_{n-2}(x_1) & \ldots & P_{n-2}(x_n) \\ P_{n+k-1}(x_1) & \ldots & \boxed{P_{n+k-1}(x_n)} \end{vmatrix} \begin{vmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ \vdots & & \vdots \\ P_{n-1}(x_1) & \ldots & \boxed{P_{n-1}(x_n)} \end{vmatrix}^{-1} .$$

We mention that there also exists a (s,d)-version of the Cauchy-Jacobi formula 7.6, obtained by substituting the (s,d)-powers $P_j(x_i)$ to the ordinary powers x_i^j.

Let us now define the (s,d)-conjugate k^l of $k \in K$ by $l \in K$ as follows

$$k^l = s(l)\,k\,l^{-1} + d(l)\,l^{-1} \ .$$

We then have the following result (*cf.* [LL]) which shows that the defining series of the elementary (s,d)-symmetric functions can be factorized in $K[X,s,d]$.

Proposition 7.12 *Let (y_i) and (z_i) be the two families of elements of K defined by*

$$z_k = \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_k \\ \vdots & & \vdots \\ P_{k-1}(x_1) & \dots & \boxed{P_{k-1}(x_k)} \end{vmatrix} \ , \quad y_k = \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_k \\ \vdots & & \vdots \\ \boxed{P_{k-1}(x_1)} & \dots & P_{k-1}(x_k) \end{vmatrix} \ .$$

Then the following relations hold in $K[X,s,d]$

$$\sum_{k \geq 0} \Lambda_{n-k}^{(s,d)}(x_1,\dots,x_n)\,(-X)^k = (X - x_n^{z_n}) \ \dots \ (X - x_1^{z_1}) = (X - x_1^{y_1}) \ \dots \ (X - x_n^{y_n}) \ .$$

Proof — We shall only establish the first identity, the second one being proved in the same way. Before going further, let us define the polynomial $V(x_1,\dots,x_n,X)$ of $K[X,s,d]$ as follows

$$V(x_1,\dots,x_n,X) = \begin{vmatrix} 1 & \dots & 1 & 1 \\ x_1 & \dots & x_n & X \\ \vdots & & \vdots & \vdots \\ P_n(x_1) & \dots & P_n(x_n) & \boxed{X^n} \end{vmatrix} \ .$$

The desired relation will then follow from the following more explicit lemma.

Lemma 7.13 *The following relation holds in $K[X,s,d]$:*

$$V(x_1,\dots,x_n,X) = V(x_2^{x_2-x_1},\dots,x_n^{x_n-x_1},X)\,(X - x_1) \ .$$

Proof — Let us first recall the following formula which holds in $K[X,s,d]$

$$X^n - P_n(k) = (X^{n-1} + \dots + X^{n-i}s(P_i(k)) + \dots + s(P_n(k)))\,(X-k) \ , \qquad (125)$$

for every $k \in K$. We also need the easily checked formula

$$(P_{n+1}(k) - P_{n+1}(l))\,(k-l)^{-1} = k^{P_n(k)-P_n(l)}\,(P_n(k) - P_n(l))\,(k-l)^{-1} + s(P_n(l)) \ , \quad (126)$$

which holds for every $k,l \in K$. Let us now apply Sylvester's noncommutative identity to the quasi-determinant $V(x_1,\dots,x_n,X)$ with the entry $(1,1)$ as pivot. We get

$$V(x_1,\dots,x_n,X) = \begin{vmatrix} \begin{vmatrix} 1 & 1 \\ x_1 & \boxed{x_2} \end{vmatrix} & \dots & \begin{vmatrix} 1 & 1 \\ x_1 & \boxed{X} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} 1 & 1 \\ P_n(x_1) & \boxed{P_n(x_2)} \end{vmatrix} & \dots & \boxed{\begin{vmatrix} 1 & 1 \\ P_n(x_1) & \boxed{X^n} \end{vmatrix}} \end{vmatrix} \ ,$$

which can be rewritten as

$$V(x_1,\dots,x_n,X) = \begin{vmatrix} x_2-x_1 & \dots & x_n-x_1 & X-x_1 \\ P_2(x_2)-P_2(x_1) & \dots & P_2(x_n)-P_2(x_1) & X^2-P_2(x_1) \\ \vdots & & \vdots & \vdots \\ P_n(x_2)-P_n(x_1) & \dots & P_n(x_n)-P_n(x_1) & \boxed{X^n-P-n(x_1)} \end{vmatrix} .$$

Using now relations (125) and (126) and basic properties of quasi-determinants, we obtain that the last quasi-determinant is equal to

$$\begin{vmatrix} 1 & \dots & 1 & 1 \\ x_2^{x_2-x_1}+s(x_1) & \dots & x_n^{x_n-x_1}+s(x_1) & X+s(x_1) \\ r_2(x_2) & \dots & r_2(x_n) & X^2+X\,s(x_1)+s(P_2(x_1)) \\ \vdots & & \vdots & \vdots \\ r_{n-1}(x_2) & \dots & r_{n-1}(x_n) & \boxed{X^{n-1}+\ldots+s(P_{n-1}(x_1))} \end{vmatrix} (X-x_1) ,$$

where we set $r_i(x_j) = x_j^{P_i(x_j)-P_i(x_1)}\,(P_i(x_j)-P_i(x_1))\,(x_j-x_1)^{-1}+s(P(P_i(x_1))$ for every i,j. Now, appropriate linear combinations of columns will cast this expression into the required form. □

Turning back to the proof of Proposition 7.12, we see that it will follow from the last lemma by induction on n. Indeed, the identity to be shown is a simple consequence of the fact that $(k^l)^m = k^{ml}$ for every $k,l,m \in K$. □

Example 7.14 For $n=2$, we have in $K[X,s,d]$

$$X^2-\Lambda_1^{(s,d)}(x_1,x_2)\,X+\Lambda_2^{(s,d)}(x_1,x_2) = (X-x_1^{x_1-x_2})\,(X-x_2) = (X-x_2^{x_2-x_1})\,(X-x_1) .$$

Thus, by expanding the right-hand side, one has

$$\Lambda_1^{(s,d)}(x_1,x_2) = x_1^{x_1-x_2}+s(x_2)+d(x_2) = x_2^{x_2-x_1}+s(x_1)+d(x_1) ,$$

$$\Lambda_2^{(s,d)}(x_1,x_2) = x_1^{x_1-x_2}\,x_2 = x_2^{x_2-x_1}\,x_1 .$$

More generally, one obtains in this way expansions of the quasi-determinantal formula of Proposition 7.9 which look like the familiar commutative expression of the elementary symmetric functions, especially in the case when $d=0$.

7.3 Polynomial symmetric functions of n noncommutative variables

In this section, we fix n noncommutative indeterminates $x_1, \dots, x_n$ and we specialize the formal symmetric functions of Section 3 by setting

$$\Lambda_k(x_1,\dots,x_n) = \sum_{i_1>i_2>\ldots>i_k} x_{i_1}\,x_{i_2}\,\dots\,x_{i_k} .$$

That is, we take,

$$\lambda(t) = \overleftarrow{\prod_{1\le k\le n}}\,(1+tx_k) = (1+tx_n)\,(1+tx_{n-1})\,(1+tx_{n-2})\,\cdots\,(1+tx_1) , \tag{127}$$

t being a commutative indeterminate. The generating series for the complete functions is thus

$$\sigma(t) = \lambda(-t)^{-1} = \overrightarrow{\prod_{1\le k\le n}} (1-tx_k)^{-1} = (1-tx_1)^{-1}\,(1-tx_2)^{-1}\,\cdots\,(1-tx_n)^{-1} \qquad (128)$$

so that the complete symmetric functions specialize to

$$S_k(x_1,\ldots,x_n) = \sum_{i_1\le i_2\le\ldots\le i_k} x_{i_1}\,x_{i_2}\,\ldots\,x_{i_k} \ .$$

The ribbon Schur functions then specialize according to the next proposition. We first introduce some vocabulary. Let $w = x_{i_1}\ldots x_{i_k}$ be a word. An integer m is called a *descent* of w if $1\le m\le k-1$ and $i_m > i_{m+1}$.

Proposition 7.15 *Let $J=(j_1,\ldots,j_n)$ be a composition of m. The ribbon Schur function R_J specializes to*

$$R_J(x_1,\ldots,x_n) = \sum x_{i_1}\,\ldots\,x_{i_m} \ , \qquad (129)$$

the sum running over all words $w = x_{i_1}\ldots x_{i_m}$ whose descent set is exactly equal to $\{j_1,\, j_1+j_2,\,\ldots\,,\, j_1+\ldots+j_{k-1}\}$.

Proof — Denote temporarily by $\overline{R}_J$ the polynomials defined by (129). It is clear that they satisfy the two relations

$$\overline{R}_{1^k} = \Lambda_k(x_1,\ldots,x_n) \ ,$$

$$\overline{R}_J\,\overline{R}_K = \overline{R}_{J\triangleright K} + \overline{R}_{J\cdot K} \ .$$

But these relations characterize the ribbon Schur functions (see Section 3.2). Therefore $\overline{R}_J = R_J(x_1,\ldots,x_n)$ for any composition J. □

Example 7.16 *Let $X=\{x_1,\, x_2,\, x_3\}$. Then*

$$\Lambda_2(X) = x_2\,x_1 + x_3\,x_1 + x_3\,x_2 \ ,$$

$$S_2(X) = x_1^2 + x_1\,x_2 + x_1\,x_3 + x_2^2 + x_2\,x_3 + x_3^2 \ ,$$

$$R_{12}(X) = x_2\,x_1^2 + x_2\,x_1\,x_2 + x_2\,x_1\,x_3 + x_3\,x_1^2 + x_3\,x_1\,x_2 + x_3\,x_1\,x_3 + x_3\,x_2^2 + x_3\,x_2\,x_3 \ ,$$

$$R_{21}(X) = x_1\,x_2\,x_1 + x_2^2\,x_1 + x_1\,x_3\,x_1 + x_2\,x_3\,x_1 + x_3^2\,x_1 + x_1\,x_3\,x_2 + x_2\,x_3\,x_2 + x_3^2\,x_2 \ .$$

The functions defined in this section are not invariant under permutation of the variables x_i. However, they are still symmetric, *i.e.* invariant under an action of the symmetric group on the free algebra $Z\langle X\rangle$ different from the usual one. This action, defined in [LS2], is compatible with Young tableaux considered as words in the free algebra. For various applications such as standard bases or Demazure's character formula see [LS3]. We recall the algorithmic description of the action of simple transpositions.

Consider first the case of a two-letter alphabet $X=\{x_1,x_2\}$. Let w be a word on X. Bracket every factor $x_2\,x_1$ of w. The letters which are not bracketed constitute a subword w_1 of w. Then, bracket every factor $x_2\,x_1$ of w_1. There remains a subword w_2.

Continue this procedure until it stops, giving a word w_k of type $w_k = x_1^r x_2^s$. The image of w under the transposition $\sigma = \sigma_{12}$ is by definition the word w^σ in which w_k is replaced by $w_k^\sigma = x_1^s x_2^r$, and the bracketed letters remain unchanged.

Let us give an example, denoting for short $w = x_{i_1} \dots x_{i_k}$ by $i_1 \dots i_k$. Choose

$$w = 1\,2\,2\,2\,1\,2\,1\,1\,1\,1\,2\,1\,2\,2\,2\ .$$

The successive bracketings of w give

$$1\,(2\,(2\,(2\,1)\,(2\,1)\,1)\,1)\,1\,(2\,1)\,2\,2\,2\ ,$$

and $w_3 = 1\,1\,2\,2\,2$. Then $w_3^\sigma = 1\,1\,1\,2\,2$ and $w^\sigma = 1\,2\,2\,2\,1\,2\,1\,1\,1\,1\,2\,1\,1\,2\,2$. Returning now to a general alphabet $X = \{x_1, x_2 \dots, x_n\}$, one defines the action of the simple transposition σ_i of x_i and x_{i+1} on the word w, by the preceding rule applied to the subword w restricted to $\{x_i, x_{i+1}\}$, the remaining letters being unchanged. For example the image by σ_2 of the word $w = 2\,1\,3\,1\,4\,2\,1\,3\,4\,3$ is $w^{\sigma_2} = 2\,1\,3\,1\,4\,2\,1\,2\,4\,3$.

It is proven in [LS2] that $w \to w^{\sigma_i}$ extends to an action of the symmetric group on $Z\langle X\rangle$, *i.e.* that given a permutation μ and a word w, all factorizations of $\mu = \sigma\sigma' \dots \sigma''$ into simple transpositions produce the same word $((w^\sigma)^{\sigma'} \dots)^{\sigma''}$ denoted w^μ. We can now state the following proposition.

Proposition 7.17 *The polynomial symmetric functions defined above are invariant under the previous action of the symmetric group.*

Proof — By definition, for any word w and any permutation μ, w^μ and w have the same descents. Therefore the ribbon Schur functions are invariant under this action, and the result follows from the fact that these functions constitute a linear basis of **Sym**. □

Denoting by $SYM(X_n)$ the algebra of polynomial symmetric functions of $X_n = \{x_1, \dots, x_n\}$, one can see that the algebra **Sym** of formal symmetric functions can be realized as the inverse limit in the category of graded algebras

$$\mathbf{Sym} \ \simeq \ \varprojlim_n \ SYM(X_n)$$

with respect to the projections $F(x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q}) \longmapsto F(x_1, \dots, x_p, 0, \dots, 0)$. One can thus realize **Sym** as $SYM(X)$, where X is an infinite alphabet. Note also that the homogeneous component of weight k of $SYM(X_n)$ has for dimension

$$\dim SYM_k(X_n) = \sum_{1 \le i \le n} \binom{k-1}{i}$$

which is equal to the dimension of the space of quasi-symmetric functions of weight k in n variables, and the duality between quasi-symmetric functions and noncommutative symmetric functions still holds in the case of a finite alphabet.

Note 7.18 The ribbon Schur functions defined in this section are particular cases of the noncommutative Schur polynomials defined in [LS2] as sums of tableaux in the free algebra. However these noncommutative Schur polynomials do not belong in general to the algebra (or skew field) generated by the elementary symmetric functions $\Lambda_k(X)$.

7.4 Symmetric functions associated with a matrix

In this section, we fix a matrix A of order n, with entries in a noncommutative ring. Recall first that in the commutative case, one has

$$\det(I + tA) = \sum_{k=0}^{n} \Lambda_k(\alpha)\, t^k \ , \tag{130}$$

$$\det(I - tA)^{-1} = \sum_{k\geq 0} S_k(\alpha)\, t^k \ , \tag{131}$$

$$-\frac{d}{dt}\left(\log\left(\det(I - tA)\right)\right) = \sum_{k\geq 1} \Psi_k(\alpha)\, t^{k-1} \ , \tag{132}$$

where α is the set of eigenvalues of A. Formula (131) is commonly known as MacMahon's master theorem.

Replacing in (130) the determinant by a quasi-determinant, we arrive at

$$|I + tA|_{ii} = \det(I + tA)\left(\det(I + tA^{ii})\right)^{-1} = \sum_{k=0}^{+\infty} \Lambda_k(\alpha - \alpha^i)\, t^k \tag{133}$$

where $\alpha - \alpha^i$ denotes the difference in the sense of λ-rings of α and the set α^i of eigenvalues of A^{ii}. We decide to take identity (133) as a definition when the entries of A no longer commute.

Definition 7.19 *Let A be a matrix of order n (with entries in an arbitrary ring), and i a fixed integer between 1 and n. The elementary symmetric functions $\Lambda_k(\alpha_i)$ associated with A (and i) are defined by*

$$|I + tA|_{ii} = \sum_{k=0}^{+\infty} \Lambda_k(\alpha_i)\, t^k \ .$$

The others families of noncommutative symmetric functions of A are defined accordingly by their generating functions

$$\sum_{k\geq 0} S_k(\alpha_i)\, t^k = |I - tA|_{ii}^{-1} \ ,$$

$$\sum_{k\geq 1} \Phi_k(\alpha_i)\, t^{k-1} = -\frac{d}{dt}\log\left(|I - tA|_{ii}\right) \ ,$$

$$\sum_{k\geq 1} \Psi_k(\alpha_i)\, t^{k-1} = |I - tA|_{ii}\, \frac{d}{dt}\, |I - tA|_{ii}^{-1} \ .$$

The specializations of the elementary, complete, power sums (of first or second kind) and ribbon functions are polynomials in the entries of A, which can be combinatorially interpreted in terms of graphs.

Proposition 7.20 *Let $\mathcal{A}$ be the complete oriented graph associated with the matrix A (cf. Section 2.2). Then,*

1. *$S_k(\alpha_i)$ is the sum of all words labelling paths of length k going from i to i in $\mathcal{A}$.*

2. *$(-1)^{k-1}\,\Lambda_k(\alpha_i)$ is the sum of all words labelling simple paths of length k going from i to i in $\mathcal{A}$.*

3. *$\Psi_k(\alpha_i)$ is the sum of all words labelling paths of length k going from i to i in $\mathcal{A}$, the coefficient of each path being the length of the first return to i.*

4. *$\Phi_k(\alpha_i)$ is the sum of all words labelling paths of length k going from i to i in $\mathcal{A}$, the coefficient of each path being the ratio of k to the number of returns to i.*

5. *Let I be a composition of k. Then $(-1)^{\ell(I)-1}\,R_I(\alpha_i)$ is the sum of all words labelling paths of length k going from i to i in $\mathcal{A}$ such that every return to i occurs at some length $\bar{I}_1^\sim + \ldots + \bar{I}_j^\sim$ with $j \in [1, \ldots, r]$ (where we set $\bar{I}^\sim = (\bar{I}_j^\sim)_{j=1,r}$).*

Proof — The graphical interpretations of Λ_k and S_k follow from Proposition 2.4. The interpretation of Ψ_n results then from the formula

$$\Psi_k = \sum_{j=1}^{k} (-1)^{j-1}\, j\, \Lambda_j\, S_{k-j} \ .$$

Similarly, the interpretation of Φ_k follows from the formula

$$\Phi_k = \sum_{j=1}^{k} (-1)^{k-j}\, \frac{k}{j} \Big(\sum_{i_1+\ldots+i_j=k} \Lambda_{i_1} \ldots \Lambda_{i_j} \Big)$$

which is a particular case of Proposition 4.18. Finally, the interpretation of R_I follows from (63). $\square$

Example 7.21 For a matrix $A = (a_{ij})$ of order $n = 3$, we have

$$\Lambda_1(\alpha_1) = S_1(\alpha_1) = \Psi_1(\alpha_1) = \Phi_1(\alpha_1) = a_{11} \ ,$$

$$\Lambda_2(\alpha_1) = R_{11}(\alpha_1) = -a_{12}\,a_{21} - a_{13}\,a_{31} \ ,$$

$$S_2(\alpha_1) = R_2(\alpha_1) = a_{11}^2 + a_{12}\,a_{21} + a_{13}\,a_{31} \ ,$$

$$\Psi_2(\alpha_1) = \Phi_2(\alpha_1) = a_{11}^2 + 2\,a_{12}\,a_{21} + 2\,a_{13}\,a_{31} \ ,$$

$$\Lambda_3(\alpha_1) = R_{111}(\alpha_1) = a_{12}\,a_{22}\,a_{21} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{32}\,a_{21} + a_{13}\,a_{33}\,a_{31} \ ,$$

$$S_3(\alpha_1) = R_3(\alpha_1) = a_{11}^3 + a_{11}\,a_{12}\,a_{21} + a_{11}\,a_{13}\,a_{31} + a_{12}\,a_{21}\,a_{11} + a_{12}\,a_{22}\,a_{21}$$
$$+ a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{31}\,a_{11} + a_{13}\,a_{32}\,a_{21} + a_{13}\,a_{33}\,a_{31} \ ,$$

$$\Psi_3(\alpha_1) = a_{11}^3 + a_{11}\,a_{12}\,a_{21} + a_{11}\,a_{13}\,a_{31} + 2\,a_{12}\,a_{21}\,a_{11} + 3\,a_{12}\,a_{22}\,a_{21}$$
$$+3\,a_{12}\,a_{23}\,a_{31} + 2\,a_{13}\,a_{31}\,a_{11} + 3\,a_{13}\,a_{32}\,a_{21} + 3\,a_{13}\,a_{33}\,a_{31} \ ,$$

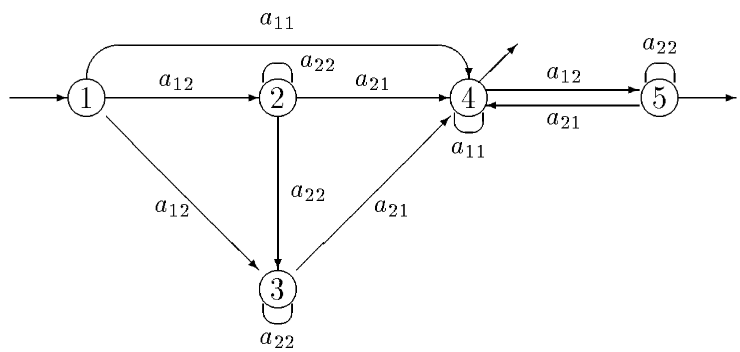
$$\Phi_3(\alpha_1) = a_{11}^3 + \frac{3}{2}\,a_{11}\,a_{12}\,a_{21} + \frac{3}{2}\,a_{11}\,a_{13}\,a_{31} + \frac{3}{2}\,a_{12}\,a_{21}\,a_{11} + 3\,a_{12}\,a_{22}\,a_{21}$$
$$+3\,a_{12}\,a_{23}\,a_{31} + \frac{3}{2}\,a_{13}\,a_{31}\,a_{11} + 3\,a_{13}\,a_{32}\,a_{21} + 3\,a_{13}\,a_{33}\,a_{31} \ ,$$

$$R_{12}(\alpha_1) = -a_{12}\,a_{21}\,a_{11} - a_{12}\,a_{22}\,a_{21} - a_{12}\,a_{23}\,a_{31} - a_{13}\,a_{31}\,a_{11} - a_{13}\,a_{32}\,a_{21} - a_{13}\,a_{33}\,a_{31} \ ,$$

$$R_{21}(\alpha_1) = -a_{11}\,a_{12}\,a_{21} - a_{11}\,a_{13}\,a_{31} - a_{12}\,a_{22}\,a_{21} - a_{12}\,a_{23}\,a_{31} - a_{13}\,a_{32}\,a_{21} - a_{13}\,a_{33}\,a_{31} \ .$$

Note 7.22 It follows from Proposition 7.20 that the generating series of the functions $S_k(\alpha_i)$, $\Lambda_k(\alpha_i)$ or $\Psi_k(\alpha_i)$ are all rational in the sense of automata theory (*cf.* Section 9). Moreover the automata which recognize these series can be explicitely described and have no multiplicities.

For example, the minimal automaton which recognizes the generating series of the functions $\Psi_k(\alpha_1)$ is given, for $n = 2$, by



On the other hand, Proposition 7.20 also shows that the generating series of the family $\Phi_k(\alpha_i)$ is not rational (*cf.* [BR] or [Eil] for more details).

7.5 Noncommutative symmetric functions and the center of $U(gl_n)$

In this section, we illustrate the results of Section 7.4 in the particular case when $A = E_n = (e_{ij})_{i,j=1,n}$, the matrix of the standard generators of the universal envelopping algebra $U(gl_n)$. Recall that these generators satisfy the relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \ .$$

The Capelli identity of classical invariant theory indicates that the determinant of the noncommutative matrix E_n makes sense, provided that one subtracts 0, 1, 2, ... $n-1$ to its leading diagonal (see [Tu] for instance). Thus one defines

$$\det E_n = \begin{vmatrix} e_{11} & e_{12} & \dots & e_{1n} \\ e_{21} & e_{22}-1 & \dots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \dots & e_{nn}-n+1 \end{vmatrix}$$

$$:= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \, e_{\sigma(1)\,1} \, (e_{\sigma(2)\,2} - \delta_{\sigma(2)\,2}) \, \dots \, (e_{\sigma(n)\,n} - (n-1)\delta_{\sigma(n)\,n}) \ .$$

This is better understood from the following well-known result (see [Ho]).

Theorem 7.23 *The coefficients of the polynomial in the variable* t

$$\det(I + tE_n) = \begin{vmatrix} 1+te_{11} & te_{12} & \dots & te_{1n} \\ te_{21} & 1+t(e_{22}-1) & \dots & te_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ te_{n1} & te_{n2} & \dots & 1+t(e_{nn}-n+1) \end{vmatrix}$$

$$:= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \, (\delta_{\sigma(1)\,1} + te_{\sigma(1)\,1}) \, \dots \, (\delta_{\sigma(n)\,n} + t(e_{\sigma(n)\,n} - (n-1)\delta_{\sigma(n)\,n})) \ .$$

generate the center $Z(U(gl_n))$ *of* $U(gl_n)$.

It is shown in [GR1] and [GR2] that the Capelli determinant can be expressed as a product of quasi-determinants. More precisely, one has

Theorem 7.24 *Keeping the notation of Theorem 7.23, $\det(I+tE_n)$ can be factorized in the algebra of formal power series in t with coefficients in $U(gl_n)$*

$$\det(I+tE_n) =$$

$$(1+te_{11})\begin{vmatrix} 1+t(e_{11}-1) & te_{12} \\ te_{21} & \boxed{1+t(e_{22}-1)} \end{vmatrix} \cdots \begin{vmatrix} 1+t(e_{11}-n+1) & \dots & te_{1n} \\ \vdots & & \vdots \\ te_{n1} & \dots & \boxed{1+t(e_{nn}-n+1)} \end{vmatrix}$$

and the factors in the right-hand side commute with each other.

Note 7.25 Theorem 7.24 is stated in [GR2] in its formal differential operator version. The above version is obtained by using the classical embedding of $U(gl_n)$ into the Weyl algebra $K[x_{ij}, \dfrac{\partial}{\partial x_{ij}}, 1 \le i, j \le n]$ of order n^2 defined by

$$e_{ij} \hookrightarrow \sum_{k=1}^{n} x_{ik} \frac{\partial}{\partial x_{jk}} .$$

We adopt the following notations. We write for short $F_m = E_m - (m-1)I = (f_{ij})_{1\le i,j\le m}$, and we put $A = F_m$, $i = m$ in Definition 7.19, the symmetric functions thus obtained being denoted by $\Lambda_k(\epsilon_m)$. In other words, by Proposition 7.20, the polynomials $\Lambda_k(\epsilon_m)$ are given by

$$\Lambda_k(\epsilon_m) = (-1)^{k-1} \sum_{1\le i_1,\dots,i_{k-1}\le m-1} f_{m\,i_1}\, f_{i_1\,i_2} \dots f_{i_{k-1} m} .$$

Combining now the two previous theorems we arrive at the following result, concerning the symmetric functions associated with the matrices E_1, $E_2 - I$, $\dots$, $E_n - (n-1)I$.

Theorem 7.26 *The symmetric functions $\Lambda_k(\epsilon_m)$ for $k \ge 0$ and $1 \le m \le n$ generate a commutative subalgebra in $U(gl_n)$. This algebra is the smallest subalgebra of $U(gl_n)$ containing the centers $Z(U(gl_1))$, $Z(U(gl_2)), \dots,$ $Z(U(gl_n))$.*

Proof — Let $\mathcal{Z}$ denote the subalgebra of $U(gl_n)$ generated by $Z(U(gl_1))$, $\dots$, $Z(U(gl_n))$. This is of course a commutative algebra. Set

$$\lambda(t,\epsilon_m) = \sum_k t^k \Lambda_k(\epsilon_m) = \begin{vmatrix} 1+t(e_{11}-m+1) & \dots & te_{1m} \\ \vdots & \ddots & \vdots \\ te_{m1} & \dots & \boxed{1+t(e_{mm}-m+1)} \end{vmatrix} .$$

By Theorem 7.23 and Theorem 7.24, one sees that the coefficients of the formal power series in t

$$\lambda_m(t) = \lambda(t,\epsilon_1)\,\lambda(t,\epsilon_2) \dots \lambda(t,\epsilon_m)$$

generate the center $Z(U(gl_m))$. Therefore the symmetric functions $\Lambda_k(\epsilon_m)$ with $k \ge 0$ and $1 \le m \le n$ generate a subalgebra of $U(gl_n)$ containing $\mathcal{Z}$. Conversely, the polynomial $\lambda_{m-1}(t)$ being invertible in $\mathcal{Z}[[t]]$, there holds

$$\lambda(t,\epsilon_m) = \lambda_{m-1}(t)^{-1}\,\lambda_m(t) ,$$

Hence the symmetric functions $\Lambda_k(\epsilon_m)$ belong to $\mathcal{Z}$, which proves the theorem. $\square$

As a consequence, the center of $U(gl_n)$ may be described in various ways in terms of the different types of symmetric functions associated with the matrices E_1, $E_2 - I$, ..., $E_n - (n-1)I$.

Corollary 7.27 *The center $Z(U(gl_n))$ is generated by the scalars and either of the following families of formal symmetric functions*

1. $\Lambda_k^{(n)} = \sum_{i_1+\ldots+i_n=k} \Lambda_{i_1}(\epsilon_1) \ldots \Lambda_{i_n}(\epsilon_n)$, *for* $1 \le k \le n$,

2. $S_k^{(n)} = \sum_{i_1+\ldots+i_n=k} S_{i_1}(\epsilon_1) \ldots S_{i_n}(\epsilon_n)$, *for* $1 \le k \le n$,

3. $\Psi_k^{(n)} = \sum_{1\le m\le n} \Psi_k(\epsilon_m)$, *for* $1 \le k \le n$.

Proof — The functions $\Lambda_k^{(n)}$ are nothing else but the coefficients of the polynomial in t $\det(I + tE_n)$, which generate $Z(U(gl_n))$ according to Theorem 7.23. Since we are in fact in a commutative situation, the functions $S_k^{(n)}$ and $\Psi_k^{(n)}$ are related to the functions $\Lambda_k^{(n)}$ by the basic identities (3) and (5), and thus also generate $Z(U(gl_n))$. $\square$

Example 7.28 The center of $U(gl_3)$ is generated by 1, $\Psi_1^{(3)}$, $\Psi_2^{(3)}$, $\Psi_3^{(3)}$, where

$$\Psi_1^{(3)} = e_{11} + e_{22} - 1 + e_{33} - 2 \ ,$$

$$\Psi_2^{(3)} = e_{11}^2 + (e_{22}-1)^2 + 2\,e_{21}\,e_{12} + (e_{33}-2)^2 + 2\,e_{31}\,e_{13} + 2\,e_{32}\,e_{23} \ ,$$

$$\begin{aligned}\Psi_3^{(3)} = e_{11}^3 &+ (e_{22}-1)^3 + (e_{22}-1)\,e_{21}\,e_{12} + 2\,e_{21}\,e_{12}\,(e_{22}-1) + 3\,e_{21}\,(e_{11}-1)\,e_{12} \\ &+(e_{33}-2)^3 + (e_{33}-2)\,e_{31}\,e_{13} + (e_{33}-2)\,e_{32}\,e_{23} + 2\,e_{31}\,e_{13}\,(e_{33}-2) + 2\,e_{32}\,e_{23}\,(e_{33}-2) \\ &+3\,e_{31}\,e_{12}\,e_{23} + 3\,e_{32}\,e_{21}\,e_{13} + 3\,e_{31}\,(e_{11}-2)\,e_{13} + 3\,e_{32}\,(e_{22}-2)\,e_{23} \ .\end{aligned}$$

The results of this section are strongly connected with Gelfand-Zetlin bases for the complex Lie algebra gl_n. Recall that each irreducible finite-dimensional gl_n-module V has a canonical decomposition into a direct sum of one-dimensional subspaces associated with the chain of subalgebras

$$gl_1 \subset gl_2 \subset \ldots \subset gl_n \ .$$

These subspaces V_μ are parametrized by Gelfand-Zetlin schemes $\mu = (\mu_{ij})_{1\le j\le i\le n}$ in such a way that for every $m = 1, 2, \ldots, n$, the module V_μ is contained in an irreducible gl_m-module with highest weight $(\mu_{m1}, \mu_{m2}, \ldots, \mu_{mm})$. Since each irreducible representation of gl_{m-1} appears at most once in the restriction to gl_{m-1} of an irreducible representation of gl_m, this set of conditions defines V_μ uniquely (*cf.* [GZ]). Moreover, the integers μ_{ij} must satisfy the condition $\mu_{ij} \ge \mu_{i-1\,j} \ge \mu_{i\,j+1}$ for all i, j.

Another characterization of the subspaces V_μ given by Cherednik [Ch] (see also Nazarov and Tarasov [NT]) is the following. The V_μ are exactly the eigenspaces of the smallest subalgebra $\mathcal{Z}$ of $U(gl_n)$ containing the centers $Z(U(gl_1))$, ..., $Z(U(gl_n))$. Taking into account Theorem 7.26, this description translates into the following proposition.

Proposition 7.29 *Let x_μ be a non-zero vector of the one-dimensional subspace V_μ parametrized by the Gelfand-Zetlin scheme $\mu = (\mu_{ij})_{1\le j\le i\le n}$. Set $\nu_{ij} = \mu_{ij} - j + 1$ and $\nu_i = \{\nu_{i1}, \ldots, \nu_{ii}\}$ (with $\nu_0 = \emptyset$). For every $F \in$ **Sym** and every $m = 1, 2, \ldots, n$, x_μ is an eigenvector of the noncommutative symmetric function $F(\epsilon_m)$. The corresponding eigenvalue is the specialization of F to the alphabet of commutative variables $\nu_m - \nu_{m-1}$ (in the sense of λ-rings) :*

$$F(\epsilon_m)\, x_\mu = F(\nu_m - \nu_{m-1})\, x_\mu \ .$$

The eigenvalues associated with different μ being pairwise distinct, this characterizes completely the subspaces V_μ.

Proof — Since the Ψ_k generate **Sym**, it is enough to prove the proposition in the case when $F = \Psi_k$. It results from Proposition 7.20 that

$$\Psi_k(\epsilon_m) =$$

$$\sum_{1\le i_1,\ldots,i_{k-1}\le m} \alpha_{i_1\ldots i_{k-1}}\,(e_{mi_1} - (m-1)\delta_{mi_1})\,(e_{i_1i_2} - (m-1)\delta_{i_1i_2})\ldots(e_{i_{k-1}m} - (m-1)\delta_{i_{k-1}m})\,,$$

where the $\alpha_{i_1\ldots i_{k-1}}$ are integers and $\alpha_{m\ldots m} = 1$. Thus, denoting by $\mathcal{N}_m$ the subalgebra of gl_m spanned by the elements e_{ij}, $1 \le i < j \le m$, we see that

$$\sum_{1\le j\le m} \Psi_k(\epsilon_j) \ - \ \sum_{1\le j\le m} (e_{jj} - j + 1)^k \ \in U(gl_m)\,\mathcal{N}_m \ . \tag{134}$$

By Corollary 7.27, $\sum_{1\le j\le m} \Psi_k(\epsilon_j)$ belongs to $Z(U(gl_m))$, and therefore acts as a scalar on any irreducible gl_m-module. By definition, x_μ belongs to the irreducible gl_m-module with highest weight $(\mu_{m1}, \mu_{m2}, \ldots, \mu_{mm})$, and we can compute this scalar by acting on the highest weight vector, which is annihilated by $\mathcal{N}_m$. Therefore, we deduce from (134) that

$$\sum_{1\le j\le m} \Psi_k(\epsilon_j)\, x_\mu = \sum_{1\le j\le m} (\mu_{mj} - j + 1)^k\, x_\mu = \Psi_k(\nu_m)\, x_\mu \ ,$$

and by difference,

$$\Psi_k(\epsilon_m)\, x_\mu = (\Psi_k(\nu_m) - \Psi_k(\nu_{m-1}))\ x_\mu = \Psi_k(\nu_m - \nu_{m-1})\, x_\mu \ .$$

□

Example 7.30 Let V be the irreducible gl_3-module with highest weight $(5, 3, 2)$. Consider the GZ-scheme

$$\mu = \begin{matrix} 5 & & 3 & & 2 \\ & 4 & & 2 & \\ & & 3 & & \end{matrix}$$

The action of the operator

$$\Psi_3(\epsilon_3) = (e_{33} - 2)^3 + (e_{33} - 2)\, e_{31}\, e_{13} + (e_{33} - 2)\, e_{32}\, e_{23} + 2\, e_{31}\, e_{13}\, (e_{33} - 2)$$

$$+2\, e_{32}\, e_{23}\, (e_{33} - 2) + 3\, e_{31}\, e_{12}\, e_{23} + 3\, e_{32}\, e_{21}\, e_{13} + 3\, e_{31}\, (e_{11} - 2)\, e_{13} + 3\, e_{32}\, (e_{22} - 2)\, e_{23}$$

on the weight vector x_μ is given by

$$\Psi_3(\epsilon_3)\, x_\mu = \left[5^3 + (3-1)^3 + (2-2)^3 - 4^3 - (2-1)^3 \right] x_\mu = 68\, x_\mu \ .$$

7.6 Twisted symmetric functions

Let K be a skew field, let s be an automorphism of K and let $(S_i^{(s)})_{i\geq 0}$ be a family of K. Let then $K[X,s]$ be the K-algebra of twisted polynomials over K, i.e. of polynomials with coefficients in K where the variable X satisfies to

$$X\,k = s(k)\,X \tag{135}$$

for every $k \in K$. Let now $\sigma(t)$ be the formal power series defined by

$$\sigma(t) = \sum_{i=0}^{+\infty} (S_i^{(s)}\,X^i)\,t^i \ .$$

The symmetric functions $S_n = S_i^{(s)}\,X^i$ defined using this generating function belong to $K[X,s]$. It is however more interesting to define twisted symmetric functions which belong to the smallest s-stable algebra generated by the family $S_i^{(s)}$.

Definition 7.31 *The* twisted symmetric functions $S_n^{(s)}$, $\Lambda_n^{(s)}$, $\Phi_n^{(s)}$, $\Psi_n^{(s)}$, $R_I^{(s)}$ *are the elements of K defined by*

$$S_n^{(s)} = S_n\,X^{-n}\ ,\ \Lambda_n^{(s)} = \Lambda_n\,X^{-n}\ ,\ \Phi_n^{(s)} = \Phi_n\,X^{-n}\ ,\ \Psi_n^{(s)} = \Psi_n\,X^{-n}\ ,\ R_I^{(s)} = R_I\,X^{-|I|}\,.$$

The quasi-determinantal formulas of Section 3.1 can be easily adapted to take care of the present situation.

Proposition 7.32

$$\Lambda_n^{(s)} = (-1)^{n-1} \begin{vmatrix} S_1^{(s)} & S_2^{(s)} & \dots & S_{n-1}^{(s)} & \boxed{S_n^{(s)}} \\ s(S_0^{(s)}) & s(S_1^{(s)}) & \dots & s(S_{n-2}^{(s)}) & s(S_{n-1}^{(s)}) \\ 0 & s^2(S_0^{(s)}) & \dots & s^2(S_{n-3}^{(s)}) & s^2(S_{n-2}^{(s)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s^{n-1}(S_0^{(s)}) & s^{n-1}(S_1^{(s)}) \end{vmatrix}\ , \tag{136}$$

$$S_n^{(s)} = (-1)^{n-1} \begin{vmatrix} \Lambda_1^{(s)} & \Lambda_2^{(s)} & \dots & \Lambda_{n-1}^{(s)} & \boxed{\Lambda_n^{(s)}} \\ s(\Lambda_0^{(s)}) & s(\Lambda_1^{(s)}) & \dots & s(\Lambda_{n-2}^{(s)}) & s(\Lambda_{n-1}^{(s)}) \\ 0 & s^2(\Lambda_0^{(s)}) & \dots & s^2(\Lambda_{n-3}^{(s)}) & s^2(\Lambda_{n-2}^{(s)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s^{n-1}(\Lambda_0^{(s)}) & s^{n-1}(\Lambda_1^{(s)}) \end{vmatrix}\ , \tag{137}$$

$$\Psi_n^{(s)} = (-1)^{n-1} \begin{vmatrix} s^{n-1}(S_1^{(s)}) & s^{n-1}(S_0^{(s)}) & 0 & \dots & 0 \\ s^{n-2}(2\,S_2^{(s)}) & s^{n-2}(S_1^{(s)}) & s^{n-2}(S_0^{(s)}) & \dots & 0 \\ s^{n-3}(3\,S_3^{(s)}) & s^{n-3}(S_2^{(s)}) & s^{n-3}(S_1^{(s)}) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boxed{n\,S_n^{(s)}} & S_{n-1}^{(s)} & S_{n-2}^{(s)} & \dots & S_1^{(s)} \end{vmatrix}\ . \tag{138}$$

Proof — We only give the proof of the first formula, the other relations being proved in the same way. According to Corollary 3.6, we have

$$\Lambda_n^{(s)}\,X^n = (-1)^{n-1}\begin{vmatrix} S_1^{(s)}\,X & S_2^{(s)}\,X^2 & \dots & S_{n-1}^{(s)}\,X^{n-1} & \boxed{S_n^{(s)}\,X^n} \\ S_0^{(s)} & S_1^{(s)}\,X & \dots & S_{n-2}^{(s)}\,X^{n-2} & S_{n-1}^{(s)}\,X^{n-1} \\ 0 & S_0^{(s)} & \dots & S_{n-3}^{(s)}\,X^{n-3} & S_{n-2}^{(s)}\,X^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & S_0^{(s)} & S_1^{(s)}\,X \end{vmatrix}.$$

Multiplying on the left the k-th row by X^{k-1} and using relation (135), we get

$$\Lambda_n^{(s)}\,X^n = (-1)^{n-1}\begin{vmatrix} S_1^{(s)}\,X & S_2^{(s)}\,X^2 & \dots & S_{n-1}^{(s)}\,X^{n-1} & \boxed{S_n^{(s)}\,X^n} \\ s(S_0^{(s)})\,X & s(S_1^{(s)})\,X^2 & \dots & s(S_{n-2}^{(s)})\,X^{n-1} & s(S_{n-1}^{(s)})\,X^n \\ 0 & s^2(S_0^{(s)})\,X^2 & \dots & s^2(S_{n-3}^{(s)})\,X^{n-1} & s^2(S_{n-2}^{(s)})\,X^n \quad : \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s^{n-1}(S_0^{(s)})\,X^{n-1} & s^{n-1}(S_1^{(s)})\,X^n \end{vmatrix}.$$

Multiplying on the right the k-th column by X^{-k}, we obtain (136). □

Using the same method, one can also give suited versions of the other determinantal identities given in Section 3. It is also interesting to notice that one can generalize these results to the case when the variable t satisfies a relation of the form

$$t\,k = s(k)\,t + d(k)$$

for every scalar $k \in K$, s being an automorphism of K and d a s-derivation. In such a case, one must consider that the S_i are the coefficients of the Laurent series $\sum_{i\geq 0}\ S_i\,t^{-i}$. In this framework, it is then easy to adapt the results given here, using formulas which can be found in [LL] or [Kr] for instance.

8 Noncommutative rational power series

In this section, we demonstrate that noncommutative symmetric functions provide an efficient tool for handling formal power series in one variable over a skew field. Special attention is paid to rational power series, and to the problem of approximating a given noncommutative power series $F(t)$ by a rational one, the so-called Padé approximation (Section 8.2). Particular Padé approximants can be obtained by expanding $F(t)$ into a noncommutative continued fraction of J-type or of S-type (Sections 8.1 and 8.4). The sequence of denominators of the partial quotients of a J-fraction is orthogonal with respect to an appropriate noncommutative scalar product, and it satisfies a three-term recurrence relation (Section 8.3). The systematic use of quasi-Schur functions enables one to derive all these results in a straightforward and unified way.

These topics gave rise to an enormous literature in the past twenty years, as shown by the 250 titles compiled by Draux in his commented bibliography [Dr]. We shall indicate only a few of them, the reader being referred to this source for precise attributions and bibliographical informations.

Interesting examples of noncommutative rational power series are provided by the generating series $\sigma(t, \alpha_i)$ of the complete symmetric functions associated with the generic matrix of order n defined in Section 7.4. Their denominators appear as noncommutative analogs of the characteristic polynomial, for which a version of the Cayley-Hamilton theorem can be established. In particular, the generic matrix posesses n *pseudo-determinants*, which are true noncommutative analogs of the determinant. These pseudo-determinants reduce in the case of $U(gl_n)$ to the Capelli determinant, and in the case of the quantum group $GL_q(n)$, to the quantum determinant (up to a power of q).

8.1 Noncommutative continued S-fractions

Continued fractions with coefficients in a noncommutative algebra have been considered by several authors, and especially by Wynn (*cf.* [Wy]). The convergence theory of these noncommutative expansions is discussed for example in [Fa]. As in the commutative case, many formulas can be expressed in terms of quasi-determinants. In this section, we give the noncommutative analog of Stieltjes continued fractions in terms of quasi-Schur functions, and specialize them to the noncommutative tangent defined in Section 5.4.2.

Let $a = (a_i)_{i\geq 0}$ be a sequence of noncommutative indeterminates. We can associate with it two types of continued S-fractions, which are the series defined by

$$a_0 \cfrac{1}{1 + a_1 t \cfrac{1}{\ddots \cfrac{1}{1 + a_n t \cfrac{1}{\ddots}}}} = a_0 \, (1 + a_1 \, t(1 + \ldots + a_{n-1} \, t(1 + a_n \, t(1 + \ldots)^{-1})^{-1} \ldots)^{-1} \,)^{-1} \tag{139}$$

and by

$$\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\ddots\, a_n\, t+1}}\ \ddots\, a_1\, t+1}}\, a_0 = ((\ldots((\ldots+1)^{-1}a_n\, t+1)^{-1}a_{n-1}\, t+\ldots+1)^{-1}a_1\, t+1)^{-1}\, a_0\ .$$

The partial fractions of these infinite continued fractions admit quasi-determinantal expressions [GR2]. Thus, the n-th convergent of (139) is equal to

$$a_0 \begin{vmatrix} \boxed{1} & a_1t & 0 & \ldots & 0 \\ -1 & 1 & a_2t & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 1 & a_nt \\ 0 & \ldots & 0 & -1 & 1 \end{vmatrix}^{-1} = a_0 \begin{vmatrix} 1 & a_2t & 0 & \ldots & \boxed{0} \\ -1 & 1 & a_3t & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 1 & a_nt \\ 0 & \ldots & 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} 1 & a_1t & 0 & \ldots & \boxed{0} \\ -1 & 1 & a_2t & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 1 & a_nt \\ 0 & \ldots & 0 & -1 & 1 \end{vmatrix}^{-1}$$

where the two quasi-determinants of the right-hand side are polynomials in t whose expansion is given by Proposition 2.6.

The following result expresses the coefficients of the noncommutative continued fraction expansion of a formal power series as *Hankel quasi-determinants*, that is, from the viewpoint of noncommutative symmetric functions, as rectangular quasi-Schur functions. Indeed, as illustrated in Section 7, any noncommutative formal power series $F(t)$ may be seen as a specialization of the generating series $\sigma(t) = \sum_{k\geq 0}\ S_k\, t^k$ of the complete symmetric functions S_k. With this in mind, the Hankel quasi-determinants associated as in the commutative case with $F(t)$ appear as the specializations of the quasi-Schur functions indexed by rectangular partitions of the form m^n.

Proposition 8.1 *Let $\sigma(t) = \sum_{k\geq 0}\ S_k\, t^k$ be a noncommutative formal power series. Then one has the following expansion of $\sigma(t)$ into a left S-fraction:*

$$\sigma(t) = \check{S}_0\ \cfrac{1}{1-\check{S}_0^{-1}\,\check{S}_1\, t\ \cfrac{1}{1+\check{S}_1^{-1}\,\check{S}_{11}\, t\ \cfrac{1}{1-\check{S}_{11}^{-1}\,\check{S}_{22}\, t\ \cfrac{1}{\ddots\ \cfrac{1}{1+(-1)^n\,\check{S}_{I_n}^{-1}\,\check{S}_{I_{n+1}}\, t\ \cfrac{1}{\ddots}}}}}}$$

where $I_{2n} = (n^n)$ and $I_{2n+1} = (n^{n+1})$.

Note 8.2 Throughout this section, we relax for convenience the assumption $S_0 = 1$ made above, and we only suppose that $S_0 \neq 0$.

Proof — Let us introduce the series $\rho(t)$ defined by

$$\sigma(t) = S_0\ \frac{1}{1-t\,\rho(t)}\ .$$

That is,

$$\rho(t) = \frac{1 - \lambda(-t)\, S_0}{t} = \sum_{n \geq 0} (-1)^{n-1}\, \Lambda_{n+1}\, S_0\, t^n \ .$$

Denote by $a_n(\sigma)$ the n-th coefficient of the Stieltjes-type development in right continued fraction of $\sigma(t)$. According to the definition of ρ,

$$a_{n+1}(\sigma) = a_n(\rho) \tag{140}$$

for $n \geq 0$. We prove by induction on n that

$$a_n(\sigma) = (-1)^n\, \check{S}_{I_n}^{-1}\, \check{S}_{I_{n+1}}$$

for every $n \geq 0$. This is clear for $n = 0,\ 1$, and the induction step is as follows. We only indicate the case $n = 2m$, the odd case being similar. According to relation (140) and to the induction hypothesis, we have

$$a_{2m+1}(\sigma) = a_{2m}(\rho) = \ \Delta_m^{-1}\, \Delta_{m+1} \tag{141}$$

where we set

$$\Delta_m = \begin{vmatrix} (-1)^{m-1}\, \Lambda_{m+1}\, S_0 & (-1)^m\, \Lambda^{m+2}\, S_0 & \dots & \boxed{\Lambda_{2m}\, S_0} \\ (-1)^m\, \Lambda_m\, S_0 & (-1)^{m-1}\, \Lambda_{m+1}\, S_0 & \dots & -\Lambda_{2m-1}\, S_0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Lambda_1\, S_0 & \Lambda_2\, S_0 & \dots & (-1)^{m-1}\, \Lambda_{m+1}\, S_0 \end{vmatrix}$$

and

$$\Delta_{m+1} = \begin{vmatrix} (-1)^{m-1}\, \Lambda_{m+1}\, S_0 & (-1)^m\, \Lambda^{m+2}\, S_0 & \dots & \boxed{-\Lambda_{2m+1}\, S_0} \\ (-1)^m\, \Lambda_m\, S_0 & (-1)^{m-1}\, \Lambda_{m+1}\, S_0 & \dots & \Lambda_{2m}\, S_0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\Lambda_1\, S_0 & \dots & (-1)^{m-1}\, \Lambda_{m+1}\, S_0 \end{vmatrix} \ .$$

Using now basic properties of quasi-determinants and Naegelbasch's formula for quasi-Schur functions, we get

$$\Delta_m = \begin{vmatrix} \Lambda_{m+1} & \Lambda_{m+2} & \dots & \boxed{\Lambda_{2m}} \\ \Lambda_m & \Lambda_{m+1} & \dots & \Lambda_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_1 & \Lambda_2 & \dots & \Lambda_{m+1} \end{vmatrix}\ S_0 = \check{S}_{m^{m+1}} = \check{S}_{I_{2m+1}} \ .$$

Arguing in the same way, we have

$$\Delta_{m+1} = - \begin{vmatrix} \Lambda_{m+1} & \Lambda_{m+2} & \dots & \boxed{\Lambda_{2m+1}} \\ \Lambda_m & \Lambda_{m+1} & \dots & \Lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \Lambda_1 & \dots & \Lambda_{m+1} \end{vmatrix}\ S_0 = \check{S}_{(m+1)^{m+1}} = \check{S}_{I_{2m+2}} \ .$$

The conclusion follows from these last two formulas and from formula (141). □

Applying ω to the formula given by Proposition 8.1, we get immediately the following one.

Corollary 8.3 *Let $\sigma(t) = \sum_{k\geq 0} S_k t^k$ be a noncommutative formal power series. Then one has the following expansion :*

$$\sigma(t) = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\ddots}\check{S}_{I_{n+1}}\check{S}_{I_n}^{-1}(-1)^n t+1}}{\ddots}\check{S}_{22}\check{S}_{11}^{-1}(-t)+1}\check{S}_{11}\check{S}_1^{-1}t+1}\check{S}_1\check{S}_0^{-1}(-t)+1}\check{S}_0$$

where $I_{2n} = (n^n)$ and $I_{2n+1} = (n^{n+1})$.

The action of ω may also be interpreted in another way in order to obtain the development in continued fraction of the inverse of a formal power series.

Corollary 8.4 *Let $\sigma(t) = \sum_{k\geq 0} S_k t^k$ be a noncommutative formal power series. Then one has the following expansion :*

$$\sigma(-t)^{-1} = \check{S}_0 \cfrac{1}{1 - \check{S}_0^{-1}\check{S}_1 t \cfrac{1}{1 + \check{S}_1^{-1}\check{S}_2 t \cfrac{1}{1 - \check{S}_2^{-1}\check{S}_{22} t \cfrac{1}{\ddots \cfrac{1}{1+(-1)^n \check{S}_{J_n}^{-1}\check{S}_{J_{n+1}} t \cfrac{1}{\ddots}}}}}}$$

where $J_{2n} = (n^n)$ and $J_{2n+1} = ((n+1)^n)$.

We apply now these results to the noncommutative tangent defined in 5.4.2. If we set $S_i(B) = T^{(r)}_{2i+1}(A)$ (A and B being merely labels to distinguish between two families of formal symmetric functions), so that

$$t\,\sigma(B;t^2) = TAN_r(A;t) = \sum_{n\geq 0} T^{(r)}_{2n+1}(A)\,t^{2n+1} \ ,$$

we can, using Propositions 8.1 and 3.21, identify the coefficients of the continued fraction expansion of the tangent in terms of staircase quasi-Schur functions. For example, one has

$$\check{S}_{333}(B) = \begin{vmatrix} T_7^{(r)} & T_9^{(r)} & \boxed{T_{11}^{(r)}} \\ T_5^{(r)} & T_7^{(r)} & T_9^{(r)} \\ T_3^{(r)} & T_5^{(r)} & T_7^{(r)} \end{vmatrix} = \begin{vmatrix} T_3^{(r)} & T_5^{(r)} & T_7^{(r)} \\ T_5^{(r)} & T_7^{(r)} & T_9^{(r)} \\ T_7^{(r)} & T_9^{(r)} & \boxed{T_{11}^{(r)}} \end{vmatrix}$$
$$= \begin{vmatrix} R_{12} & R_{122} & R_{1222} \\ R_{122} & R_{1222} & R_{12222} \\ R_{1222} & R_{12222} & \boxed{R_{122222}} \end{vmatrix} = \check{S}_{123456}(A) \ ,$$

the last equality following from Proposition 3.21.

This identity may be seen as a particular case of the following proposition, which expresses more general quasi-determinants in the $T^{(r)}_{2n+1}$ as quasi Schur-functions (indexed by skew partitions). Let ρ_k denote the staircase partition $(1, 2, \ldots, k)$.

Proposition 8.5 *Let $I = (i_1, \ldots, i_n)$ be a partition, such that $i_1 \geq n-1$. Set $N = i_n + n$. Then one has*

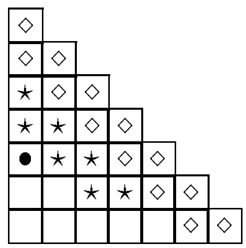
$$\check{S}_I(B) = \check{S}_{\rho_N/J}(A) \ , \tag{142}$$

where J is the partition whose diagram is obtained by removing from the diagram of ρ_N, n successive ribbon shapes indexed by the compositions

$$12^{i_n+n-1},\ 12^{i_{n-1}+n-3},\ \ldots\ , 12^{i_1-n+1} \ .$$

Proof — This follows from Bazin's theorem for quasi-determinants. $\square$

Example 8.6 Take $I = (2, 3, 4)$ so that $N = 7$. The ribbon shapes to be extracted from the diagram of ρ_7 are 12^6, 12^3, 1. Hence, $J = (2, 5)$, as illustrated by the following picture.



Thus, $\check{S}_{234}(B) = \check{S}_{1234567/25}(A)$.

We end this Section by considering the specialization

$$S_i \longrightarrow 0 \ , \quad \text{for } i > n \ .$$

In this case the continued fraction for $TAN_r(A, t)$ terminates, and by means of the recurrence formulas given in [GR2] for computing its numerator and denominator, one obtains

Proposition 8.7 *Suppose that $S_i = 0$ for $i > n$. Then the complete functions S_i, $i \leq n$ may be expressed as rational functions of the staircase quasi Schur functions S_{ρ_i}, $i \leq n$. Namely, writing for short $c_i = \check{S}_{\rho_i}^{-1}\, \check{S}_{\rho_{i+1}}$ for $1 \leq i < n$ and $c_i = 0$ for $i \geq n$, one has*

$$S_2 = \sum_{1 \leq i} c_i \ , \quad S_4 = \sum_{1 \leq i \leq j} c_i\, c_{j+2} \ , \quad S_6 = \sum_{1 \leq i \leq j \leq k} c_i\, c_{j+2}\, c_{k+4} \ , \ \ldots$$

$$S_1^{-1}\, S_3 = \sum_{2 \leq i} c_i \ , \ \mathbf{S}_1^{-1}\, S_5 = \sum_{2 \leq i \leq j} c_i\, c_{j+2} \ , \ \ \mathbf{S}_1^{-1}\, S_7 = \sum_{2 \leq i \leq j \leq k} c_i\, c_{j+2}\, c_{k+4} \ , \ \ldots$$

Proposition 8.7 may be regarded as a noncommutative analog of a classical question investigated by Laguerre and Brioschi. The problem was to express any symmetric polynomial of n indeterminates as a rational function of the power sums of odd degree Ψ_{2k+1} (*cf.* [Lag], [Po]). Foulkes gave a solution by means of Schur functions. Indeed, in the commutative case, the only Schur functions belonging to the ring $\mathbf{Q}[\Psi_1, \Psi_3, \Psi_5, \ldots]$ are the one indexed by staircase partitions ρ_n, so that Laguerre's problem is equivalent to expressing a symmetric polynomial of n indeterminates as a rational function of the staircase Schur functions (*cf.* [F3]). Note that in the noncommutative setting, this no longer holds. For example

$$\check{S}_{12} = R_{12} = -1/3\, \Psi^3 - 1/6\, \Psi^{12} + 1/6\, \Psi^{21} + 1/3\, \Psi^{111} \ ,$$

does not belong to $\mathbf{Q}[\Psi_1, \Psi_3, \Psi_5, \ldots]$. The fact that the symmetric functions depend only on n variables is replaced in Proposition 8.7 by the condition $S_i = 0$ for $i > n$.

8.2 Noncommutative Padé approximants

The classical Padé approximant is the ratio of two determinants, in which all columns, except for the first ones, are identical. In the commutative case, these determinants can be interpreted as *multi Schur functions* (see [La]). One obtains a noncommutative analog by replacing determinants by quasi-determinants. Here the problem is to approximate a noncommutative formal series

$$F(t) = S_0 + S_1\,t + S_2\,t^2 + \ \ldots + S_n\,t^n + \ \ldots \quad , \tag{143}$$

where the parameter t commutes with the S_i, by a rational fraction $Q(t)^{-1}P(t)$ up to a fixed order in t. This kind of computation appears for instance in Quantum Field Theory ([Be], [GG]), F being in this case a perturbation series, or in electric networks computations [BB].

Proposition 8.8 *Let $S_0, S_1, \ldots, S_{m+n}$ be noncommutative indeterminates, and let t be an indeterminate commuting with the S_i. We set $C_k(t) = S_0 + S_1\,t + \cdots + S_k\,t^k$ ($C_k(t) = 0$ if $k < 0$), and*

$$P_m(t) = \begin{vmatrix} \boxed{C_m(t)} & S_{m+1} & \cdots & S_{m+n} \\ t\,C_{m-1}(t) & S_m & \cdots & S_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t^n\,C_{m-n}(t) & S_{m-n+1} & \cdots & S_m \end{vmatrix} , \tag{144}$$

$$Q_n(t) = \begin{vmatrix} \boxed{1} & S_{m+1} & \cdots & S_{m+n} \\ t & S_m & \cdots & S_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t^n & S_{m-n+1} & \cdots & S_m \end{vmatrix} . \tag{145}$$

Then we have

$$Q_n(t)\cdot(S_0 + S_1\,t + \cdots + S_{m+n}\,t^{m+n}) = P_m(t) + O(t^{m+n+1}) \ .$$

Proof — Expanding $Q_n(t)$ by its first column and multiplying to the right by $S_0 + S_1\,t + \cdots + S_{m+n}\,t^{m+n}$, one obtains for the terms in t^k with $k \le m+n$ the following expression :

$$\begin{vmatrix} \boxed{S_{m+n}} & S_{m+1} & \cdots & S_{m+n} \\ S_{m+n-1} & S_m & \cdots & S_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_m & S_1 & \cdots & S_m \end{vmatrix} t^{m+n} + \begin{vmatrix} \boxed{S_{m+n-1}} & S_{m+1} & \cdots & S_{m+n} \\ S_{m+n-2} & S_m & \cdots & S_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m-1} & S_1 & \cdots & S_m \end{vmatrix} t^{m+n-1} + \cdots$$

$$+ \cdots + \begin{vmatrix} \boxed{S_0} & S_{m+1} & \cdots & S_{m+n} \\ 0 & S_m & \cdots & S_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & S_1 & \cdots & S_m \end{vmatrix} t^0 = P_m(t) \ ,$$

since the coefficients of the t^k with $k > m$ are zero, the corresponding quasi-determinants having two identical columns. □

Definition 8.9 *The Padé approximants of the noncommutative series* (143) *are the rational functions*

$$[m/n] = Q_n(t)^{-1}\, P_m(t) \ .$$

We have for instance

$$[1/1] = (1 - S_2\, S_1^{-1}\, t)^{-1}\, (S_0 + (S_1 - S_2\, S_1^{-1} S_0)\, t) = S_0 + S_1\, t + S_2\, t^2 + O(t^3) \ .$$

Applying the $*$ involution (*cf.* 3.1, 3.3, and 8.3 below), one obtains the equivalent expression $[m/n] = R_m(t)\, T_n(t)^{-1}$, where

$$R_m(t) = \begin{vmatrix} \boxed{C_m(t)} & tC_{m-1}(t) & \dots & t^n C_{m-n}(t) \\ S_{m+1} & S_m & \dots & S_{m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n} & S_{m+n-1} & \dots & S_m \end{vmatrix} \ , \tag{146}$$

$$T_n(t) = \begin{vmatrix} \boxed{1} & t & \dots & t^n \\ S_{m+1} & S_m & \dots & S_{m-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m+n} & S_{m+n-1} & \dots & S_m \end{vmatrix} \ . \tag{147}$$

Thus one also has

$$[1/1] = (S_0 + (S_1 - S_0 S_1^{-1} S_2)t)(1 - S_1^{-1} S_2 t)^{-1} \ .$$

This last remark may be interpreted as an effective proof of the fact that the ring of polynomials in one indeterminate with coefficients in a skew field is an Ore domain. For, given two polynomials $P(t)$ and $Q(t)$ of respective degrees m and n with no common left factor, and assuming $Q(0) \neq 0$, one has

$$P(t)\, T(t) = Q(t)\, R(t) \ ,$$

where $R(t)\, T(t)^{-1}$ is the Padé approximant $[m/n]$ of the power series $Q(t)^{-1}\, P(t)$.

8.3 Noncommutative orthogonal polynomials

Motivated by formula (145), we consider in this section the sequence of polynomials

$$\pi_n(x) = \begin{vmatrix} S_n & \cdots & S_{2n-1} & \boxed{x^n} \\ S_{n-1} & \cdots & S_{2n-2} & x^{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & 1 \end{vmatrix} \ , \tag{148}$$

in the commutative variable x. In the case when the S_i commute with each other, $\pi_n(x)$ is none other than the sequence of orthogonal polynomials associated with the moments S_i. Thus, when the S_i belong to a skew field, these polynomials may be regarded as natural noncommutative analogs of orthogonal polynomials. They are indeed orthogonal with respect to the noncommutative scalar product to be defined now. This scalar product is a formal analog of the matrix-valued ones used for example in [GG] and [BB].

Let R be the ring of polynomials in the commutative indeterminate x, with coefficients in the free field $K \not< S_0, S_1, S_2, \dots \not>$. Recall that this field is endowed with a natural involution $F \longrightarrow F^*$ defined as the *anti*-automorphism such that $S_i^* = S_i$ for all i. This involution is extended to R by putting

$$\left(\sum_i a_i\, x^i\right)^* = \sum_i a_i^*\, x^i \ .$$

In particular, we have

$$\pi_n(x)^* = \begin{vmatrix} S_{n-1} & S_n & \cdots & S_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_0 & S_1 & \cdots & S_n \\ 1 & x & \cdots & \boxed{x^n} \end{vmatrix} \ .$$

We define a scalar product $\langle . \, , . \rangle$ in R by

$$\langle \sum_i a_i\, x^i \, , \, \sum_j b_j\, x^j \rangle := \sum_{i,j} a_i\, S_{i+j}\, b_j^* \ .$$

This product satisfies the following obvious properties

$$\langle \alpha_1\, p_1(x) + \alpha_2\, p_2(x) \, , \, q(x) \rangle = \alpha_1 \, \langle p_1(x) \, , \, q(x) \rangle + \alpha_2 \, \langle p_2(x) \, , \, q(x) \rangle \ ,$$

$$\langle p(x) \, , \, \beta_1\, q_1(x) + \beta_2\, q_2(x) \rangle = \langle p(x) \, , \, q_1(x) \rangle \, \beta_1^* + \langle p(x) \, , \, q_2(x) \rangle \, \beta_2^* \ ,$$

$$\langle q(x) \, , \, p(x) \rangle = \langle p(x) \, , \, q(x) \rangle^* \ ,$$

$$\langle x\, p(x) \, , \, q(x) \rangle = \langle p(x) \, , \, x\, q(x) \rangle \ ,$$

from which one deduces, by means of 2.12, that

$$\langle \pi_n(x) \, , \, x^i \rangle = \check{S}_{n^n i} \ , \qquad \langle x^i \, , \, \pi_n(x) \rangle = \check{S}_{n^n i}^* \ ,$$

for $i \geq n$, and that

$$\langle \pi_n(x) \, , \, x^i \rangle = \langle x^i \, , \, \pi_n(x) \rangle = 0$$

for $i = 0, 1, \dots , n-1$. Hence we obtain

Proposition 8.10 *The sequence of polynomials $\pi_n(x)$ satisfies $\langle \pi_n(x) \, , \, \pi_m(x) \rangle = 0$ for $n \neq m$.* $\square$

As in the commutative case, the sequence $\pi_n(x)$ follows a typical three term recurrence relation that we shall now explicit. Set

$$x\, \pi_n(x) = \sum_{0 \leq k \leq n+1} a_k^{(n)}\, \pi_k(x) \ .$$

It follows from 8.10 that

$$\langle x\, \pi_n(x) \, , \, \pi_k(x) \rangle = a_k^{(n)} \langle \pi_k(x) \, , \, \pi_k(x) \rangle = a_k^{(n)} \langle \pi_k(x) \, , \, x^k \rangle = a_k^{(n)}\, \check{S}_{k^{k+1}} \ .$$

On the other hand, if $k \leq n-2$,

$$\langle x\, \pi_n(x) \, , \, \pi_k(x) \rangle = \langle \pi_n(x) \, , \, x\, \pi_k(x) \rangle = 0$$

and $a_k^{(n)} = 0$ for $k = 0, 1, \ldots, n-2$. The polynomials $\pi_k(x)$ being monic, it is clear that $a_{n+1}^{(n)} = 1$, and it remains only to compute $a_n^{(n)}$ and $a_{n-1}^{(n)}$. We have

$$\langle x\,\pi_n(x)\,,\,\pi_{n-1}(x)\rangle = \langle \pi_n(x)\,,\,x\,\pi_{n-1}(x)\rangle = \langle \pi_n(x)\,,\,x^n\rangle = \check{S}_{n^{n+1}}\ ,$$

and, also, by expanding the quasi-determinant $\pi_n(x)$ by its last column,

$$\langle x\,\pi_n(x)\,,\,\pi_n(x)\rangle = \langle x^{n+1}, \pi_n(x)\rangle - \check{S}^*_{(n-1)^{n-1}n}\,\check{S}^{-1}_{(n-1)^n}\langle x^n, \pi_n(x)\rangle$$

$$= \check{S}^*_{n^n(n+1)} - \check{S}^*_{(n-1)^{n-1}n}\,\check{S}^{-1}_{(n-1)^n}\,\check{S}_{n^{n+1}}\ .$$

Hence we obtain

Proposition 8.11 *The noncommutative orthogonal polynomials $\pi_n(x)$ satisfy the three term recurrence relation*

$$\pi_{n+1}(x) - (x - \check{S}^*_{n^n(n+1)}\check{S}^{-1}_{n^{n+1}} + \check{S}^*_{(n-1)^{n-1}n}\,\check{S}^{-1}_{(n-1)^n})\pi_n(x) + \check{S}_{n^{n+1}}\,\check{S}^{-1}_{(n-1)^n}\,\pi_{n-1}(x) = 0 \quad (149)$$

for $n \geq 1$. □

Applying the $*$ involution to (149), we get a similar recurrence for the polynomials $\pi_n^*(x)$, namely

$$\pi^*_{n+1}(x) - \pi^*_n(x)\,(x - \check{S}^{-1}_{n^{n+1}}\,\check{S}_{n^n(n+1)} + \check{S}^{-1}_{(n-1)^n}\,\check{S}_{(n-1)^{n-1}n}) + \check{S}^{-1}_{(n-1)^n}\,\check{S}_{n^{n+1}}\,\pi_{n-1}(x) = 0 \quad (150)$$

Note also that the symmetric function $\check{S}^*_{n^n(n+1)} - \check{S}^*_{(n-1)^{n-1}n}\,\check{S}^{-1}_{(n-1)^n}\,\check{S}_{n^{n+1}}$ being equal to the scalar product $\langle x\,\pi_n(x)\,,\,\pi_n(x)\rangle$, it is invariant under the $*$ involution, which gives the following interesting identity

$$\check{S}^*_{n^n(n+1)} - \check{S}^*_{(n-1)^{n-1}n}\,\check{S}^{-1}_{(n-1)^n}\,\check{S}_{n^{n+1}} = \check{S}_{n^n(n+1)} - \check{S}_{n^{n+1}}\,\check{S}^{-1}_{(n-1)^n}\check{S}_{(n-1)^{n-1}n}\ .$$

8.4 Noncommutative continued J-fractions

In this section we consider continued fractions of the type

$$b_1\,\cfrac{1}{x - a_1 + b_2\,\cfrac{1}{x - a_2 + b_3\,\cfrac{1}{\ddots\,\cfrac{1}{x - a_n + b_{n+1}\,\cfrac{1}{\ddots}}}}} \qquad (151)$$

In the commutative case, these fractions are called J-fractions, and their connexion with orthogonal polynomials and Padé approximants is well-known (*cf.* [Wa]). As observed by several authors (see *e.g.* [GG] [BB], and [Dr] for other references) similar properties hold true when (a_i) and (b_i) are replaced by two sequences of noncommutative indeterminates.

Denote by $p_n(x)\,q_n(x)^{-1}$ the n-th partial quotient of (151):

$$p_n(x)\,q_n(x)^{-1} = b_1 \cfrac{1}{x - a_1 + b_2 \cfrac{1}{x - a_2 + b_3 \cfrac{1}{\ddots \cfrac{1}{x - a_n}}}}$$

It is convenient to put also $p_0 = 0$, $q_0 = 1$. The polynomials $p_n(x)$ and $q_n(x)$ then satisfy the three term recurrence relation [GR2], [BR]

$$p_{n+1} = p_n\,(x - a_{n+1}) + p_{n-1}\,b_{n+1}\ , \qquad q_{n+1} = q_n\,(x - a_{n+1}) + q_{n-1}\,b_{n+1} \tag{152}$$

Comparing with (150), we are led to the following result

Theorem 8.12 *Let S_i be a sequence of noncommutative indeterminates. The expansion of the formal power series $\sum_{k\ge 0} S_k\,x^{-k-1}$ into a noncommutative J-fraction is equal to*

$$b_1 \cfrac{1}{x - a_1 + b_2 \cfrac{1}{\ddots \cfrac{1}{x - a_n + b_{n+1} \cfrac{1}{\ddots}}}} \tag{153}$$

where $a_1 = S_1$, $b_1 = S_0$, and

$$a_n = \check{S}^{-1}_{(n-1)^n}\,\check{S}_{(n-1)^{n-1}n} - \check{S}^{-1}_{(n-2)^{n-1}}\,\check{S}_{(n-2)^{n-2}(n-1)}\ ,$$

$$b_n = \check{S}^{-1}_{(n-2)^{n-1}}\,\check{S}_{(n-1)^n}$$

for $n \ge 2$. The n-th partial quotient $p_n(x)\,q_n(x)^{-1}$ is given by

$$p_n(x) = \begin{vmatrix} S_{n-1} & \cdots & S_{2n-2} & S_{2n-1} \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & S_n \\ 0 & \cdots & \sum_{k=0}^{n-2} S_k\,x^{n-k-2} & \boxed{\sum_{k=0}^{n-1} S_k\,x^{n-k-1}} \end{vmatrix}\ , \tag{154}$$

$$q_n(x) = \begin{vmatrix} S_{n-1} & \cdots & S_{2n-2} & S_{2n-1} \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & S_n \\ 1 & \cdots & x^{n-1} & \boxed{x^n} \end{vmatrix}\ , \tag{155}$$

and one has

$$p_n(x)\,q_n(x)^{-1} = S_0\,x^{-1} + \cdots + S_{2n-1}\,x^{-2n} + O(x^{-2n-1})\ .$$

Proof — This is just a matter of properly relating the results of the previous sections. We first prove that the numerator and denominator of the n-th convergent of (153) admit the quasi-determinantal expressions (154) and (155). To this aim we denote temporarily by

$\overline{p}_n(x)$, $\overline{q}_n(x)$ the right-hand sides of (154) and (155), and we check that these polynomials satisfy the recurrence relations (152). The relation for $\overline{q}_n(x)$ is none other than (150) (with the necessary change of notations). Since $\overline{q}_0 = q_0$ and $\overline{q}_1 = q_1$, we deduce that $\overline{q}_n = q_n$ for all n. The same argument is applied to $\overline{p}_n(x)$. Indeed, it results from (146), (147) and Proposition 8.8 with $t = x^{-1}$, $m = n-1$, that

$$\overline{p}_n(x) = (\sum_{k\geq 0} S_k\, x^{-k-1})\, q_n(x) + O(x^{-2n-1}) \ ,$$

which shows that $\overline{p}_n(x)$ satisfies the same recurrence relation as $q_n(x)$. Since $\overline{p}_1 = p_1$, and $\overline{p}_2 = p_2$ as one easily checks, we get that $\overline{p}_n = p_n$ for all n, and the rest of the theorem follows. □

8.5 Rational power series and Hankel matrices

Denote by $K[[t]]$ the ring of formal power series in t with coefficients in the skew field K. A series $F(t) = \sum_{k\geq 0} S_k\, t^k$ in $K[[t]]$ is said to be *rational* iff it is a rational fraction, that is, iff there exists polynomials P, Q in $K[t]$ such that $F(t) = Q(t)^{-1}\, P(t)$.

We begin by extending to the noncommutative case the classical characterization of rational series by means of Hankel determinants. With every series $F(t) = \sum_{k\geq 0} S_k\, t^k$ in $K[[t]]$ is associated an infinite Toeplitz matrix $\mathbf{S} = (S_{j-i})_{i,j\geq 0}$. The quasi-minors of $\mathbf{S}$ are conveniently seen as specializations of quasi-Schur functions and we shall still denote them by $\check{S}_I$, the series F being understood. In particular, the quasi-Schur functions indexed by rectangular partitions correspond to Hankel quasi-determinants.

Proposition 8.13 *The series $F(t)$ is rational iff $\mathbf{S}$ is of finite rank. More explicitely, $F(t) = Q^{-1}(t)\, P(t)$ where $P(t)$ and $Q(t)$ have respective degrees m and n, and no common left factor, iff the submatrix*

$$R = \begin{pmatrix} S_m & S_{m+1} & \dots & S_{m+n-1} \\ S_{m-1} & S_m & \dots & S_{m+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m-n+1} & S_{m-n+2} & \dots & S_m \end{pmatrix}$$

is invertible and $\check{S}_{(m+1)^n(m+p)} = 0$ for every $p \geq 1$. Q and P are then given by formulas (145) and (144).

Proof — Suppose that R is invertible. Then one can define Q and P by formulas (145) (144), and the same computation as in the proof of Proposition 8.8 shows that

$$Q(t)\, F(t) = P(t) + \sum_{p\geq 1} \check{S}_{(m+1)^n(m+p)}\, t^{m+n+p} \ .$$

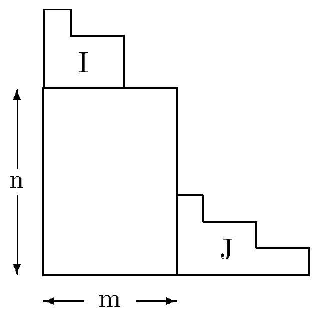
Hence, if $\check{S}_{(m+1)^n(m+p)} = 0$ for $p \geq 1$, one obtains $F(t) = Q^{-1}(t)\, P(t)$. Conversely, if $F(t)$ is rational, one can write $F(t) = Q^{-1}(t)\, P(t)$ with $Q(0) = 1$, P and Q having no common left factor, that is, P and Q of minimal degree. Therefore, if $\deg Q = n$, $\deg P = m$, we have, setting $Q(t) = \sum_{0\leq i\leq n} b_i\, t^i$,

$$\sum_{0\leq i\leq n} b_i\, S_{r-i} = 0 \ , \quad r \geq m+1 \ , \qquad (156)$$

which shows that the matrix R has rank less or equal to n. This rank cannot be less than n, otherwise P and Q would not be of minimal degree. Therefore R is invertible, and the relation $\check{S}_{(m+1)^n(m+p)} = 0$ follows from (156). $\square$

The coefficients S_i of a rational series $F(t) = Q^{-1}(t)\,P(t)$ satisfy many remarkable identities. The following result may be seen as a natural analog of the factorization formulas for Schur polynomials in two finite sets of commutative variables given in [BeRe]. The reader is referred to [Pr] for algebro-geometric applications of these important relations in the commutative case.

We denote by $(I,\, m^n + J)$ the partition whose Young diagram is

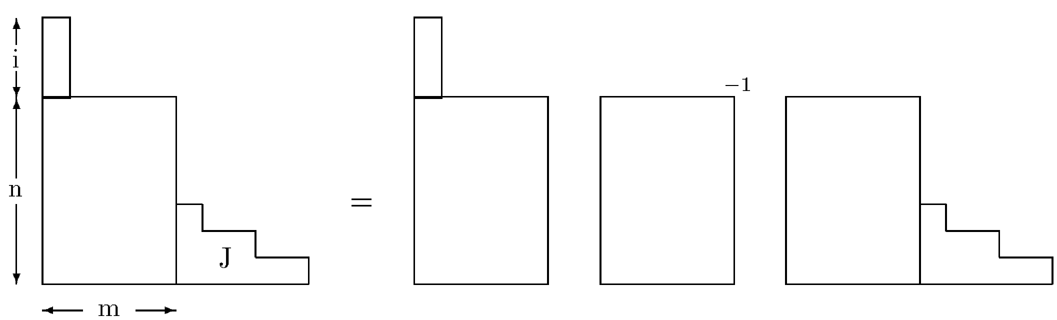


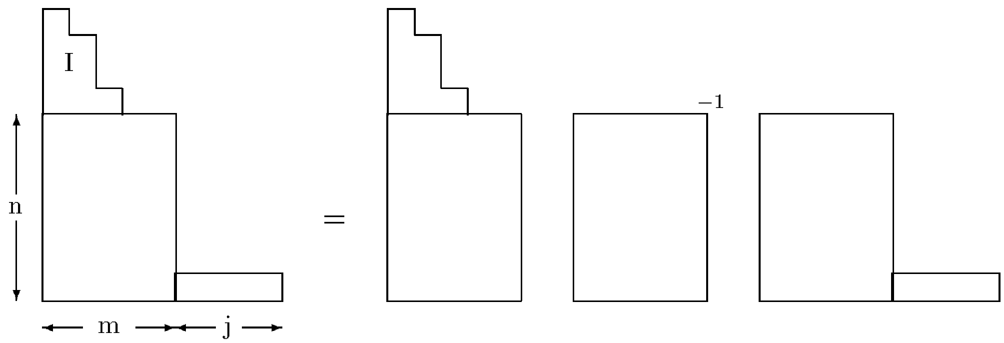
Proposition 8.14 *Let $F(t) = Q^{-1}(t)\,P(t)$ be a rational series, and assume that $P(t)$ and $Q(t)$ have respective degrees m and n, and no common left factor. Let I be a partition whose largest part is less or equal to m, J be a partition whose length is less or equal to n, and i, j be integers. There holds*

$$\check{S}_{(1^i,\, m^n+J)} = \check{S}_{(1^i,\, m^n)}\, \check{S}_{m^n}^{-1}\, \check{S}_{(m^n+J)} \ , \tag{157}$$

$$\check{S}_{(I,\, m^n+j)} = \check{S}_{(I,\, m^n)}\, \check{S}_{m^n}^{-1}\, \check{S}_{(m^n+j)} \ . \tag{158}$$

Replacing each quasi-Schur function $\check{S}_H$ by the Young diagram of H, relations (157) and (158) are pictured as follows





The proof of Proposition 8.14 is a consequence of the next Lemma, valid for generic quasi-Schur functions and of independent interest.

Lemma 8.15 *Let $H = (h_1, \ldots, h_r)$ be a partition, and j, k be two integers with $k \le h_1$. Then,*

$$\begin{vmatrix} \boxed{\check{S}_{(k,\,H+j)}} & \check{S}_{(k,\,H)} \\ \check{S}_{H+j} & \check{S}_{H} \end{vmatrix} = \check{S}_{(h_1+1,\, h_2+1,\, \ldots\, h_r+1,\, h_r+j)} \ . \qquad (159)$$

Proof — This is a simple consequence of Bazin's theorem. □

Proof of Proposition 8.14 — Let K denote the partition obtained from I by removing its first part. The hypothesis imply that, in the specialization to the series F, the quasi-Schur functions $\check{S}_{(L,\,(m+1)^n(m+p))}$ are sent to 0, for any $p \ge 1$ and any partition L whose largest part is less or equal to $m+1$. Therefore, putting $H = (K, m^n)$ in (159), we get

$$\check{S}_{(I,\,m^n+j)} = \check{S}_{(I,\,m^n)}\, \check{S}^{-1}_{(K,\,m^n)}\, \check{S}_{(K,\,m^n+j)} \ ,$$

and formula (158) follows by induction on the length of I. Formula (157) may be proved similarly, or derived directly from (158) by means of the involution ω. □

8.6 The characteristic polynomial of the generic matrix, and a noncommutative Cayley-Hamilton theorem

In this section, we illustrate the results of Section 8.5 on a fundamental example, namely we take $F(t)$ to be the generating series $\sigma(t, \alpha_i) = \sum_{k\ge 0} S_k(\alpha_i)\, t^k = |I - tA|^{-1}_{ii}$ of the complete symmetric functions associated with the generic $n \times n$ matrix A, and its i-th diagonal entry a_{ii} (*cf.* Section 7.4). We first prove that this series is rational.

Proposition 8.16 *For the generic $n \times n$ matrix $A = (a_{ij})$, there holds*

$$\sigma(t, \alpha_i) = Q_i(t)^{-1}\, P_i(t) \ ,$$

Q_i and P_i being polynomials of degree n and $n-1$ without left common factor, with coefficients in the free field generated by the entries a_{ij} of A. Moreover, assuming the normalizing condition $Q_i(0) = P_i(0) = 1$, the polynomials $Q_i(t)$, $P_i(t)$ are unique.

Proof — Let v_k denote the i-th row vector of A^k. In other words, the sequence v_k is defined by $v_0 = (0, \dots, 1, \dots, 0)$ (the entry 1 is at the i-th place) and, for $k \geq 0$, by $v_{k+1} = v_k\, A$. The v_k may be regarded as vectors in $K \not< A \not>^n$, which is at the same time a left and right n-dimensional vector space on the free field $K \not< A \not>$. Hence, there exists scalars $\lambda_0, \lambda_1, \dots, \lambda_n$ in $K \not< A \not>$ such that one has

$$\lambda_0\, v_0 + \lambda_1\, v_1 + \ldots + \lambda_n\, v_n = 0 \ . \tag{160}$$

Multiplying (160) from the right by A^j, and using the fact that the i-th component of v_k is equal to $S_k(\alpha_i)$, one obtains the relations

$$\sum_{0 \leq p \leq n} \lambda_p\, S_{p+j}(\alpha_i) = 0 \ , \quad j \geq 0 \ , \tag{161}$$

which shows that the rank of the Hankel matrix $\mathbf{S}(\alpha_i)$ is finite, equal to n. More precisely, we have,

$$\begin{vmatrix} S_n(\alpha_i) & \dots & S_{2n-1}(\alpha_i) & \boxed{S_{j+n}(\alpha_i)} \\ \vdots & \ddots & \vdots & \vdots \\ S_1(\alpha_i) & \dots & S_n(\alpha_i) & S_{j+1}(\alpha_i) \\ S_0(\alpha_i) & \dots & S_{n-1}(\alpha_i) & S_j(\alpha_i) \end{vmatrix} = 0 \ , \quad j \geq 0 \ , \tag{162}$$

and the conclusion follows from Proposition 8.13. □

Let A^{ii} denote the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and column of A. As was explained in the introduction of Section 7.4, the series $\sigma(t, \alpha_i)$ is a noncommutative analog of $\det(I - tA^{ii})/\det(I - tA)$, that is, of the ratio of the characteristic polynomial of A^{ii} to the characteristic polynomial of A (up to the change of variable $u = t^{-1}$). Therefore, we can regard the polynomials $Q_i(t)$ of Proposition 8.16 as natural analogs of $\det(I - tA)$. This is supported by the fact that these polynomials give rise to a form of the Cayley-Hamilton theorem for the generic noncommutative matrix.

For instance, when $n = 2$, the two polynomials $Q_1(t)$, $Q_2(t)$ are given by

$$Q_1(t) = 1 - (a_{11} + a_{12}\, a_{22}\, a_{12}^{-1})\, t + (a_{12}\, a_{22}\, a_{12}^{-1}\, a_{11} - a_{12}\, a_{21})\, t^2 \ , \tag{163}$$

$$Q_2(t) = 1 - (a_{22} + a_{21}\, a_{11}\, a_{21}^{-1})\, t + (a_{21}\, a_{11}\, a_{21}^{-1}\, a_{22} - a_{21}\, a_{12})\, t^2 \ . \tag{164}$$

Writing for short $Q_i(t) = 1 - \mathrm{tr}_i(A)\, t + \det_i(A)\, t^2$ for $i = 1, 2$, one can check that

$$A^2 - \begin{pmatrix} \mathrm{tr}_1(A) & 0 \\ 0 & \mathrm{tr}_2(A) \end{pmatrix} A + \begin{pmatrix} \det_1(A) & 0 \\ 0 & \det_2(A) \end{pmatrix} = 0 \ , \tag{165}$$

the Cayley-Hamilton theorem for the generic matrix of order 2. The general result is as follows. Set

$$Q_i(t) = \sum_{0 \leq j \leq n} (-1)^j\, L_j^{(i)}(A)\, t^j \ , \tag{166}$$

$$L_j(A) = \begin{pmatrix} L_j^{(1)}(A) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & L_j^{(n)}(A) \end{pmatrix} \ , \tag{167}$$

for every $1 \leq i \leq n$.

Theorem 8.17 *The generic noncommutative matrix A satisfy the following polynomial equation, the coefficients of which are diagonal matrices with entries in $K \not< A \not>$*

$$\sum_{0 \le j \le n} (-1)^j \, L_j(A) \, A^{n-j} = 0 \ . \tag{168}$$

Proof — One has to check that the i-th row vector of the left-hand side of (168) is zero. But this is exactly what is done in the proof of Proposition 8.16. Indeed, keeping the notations therein, the coefficients $L_j^{(i)}(A)$ of

$$Q_i(t) = \begin{vmatrix} S_n(\alpha_i) & \dots & S_{2n-1}(\alpha_i) & \boxed{1} \\ \vdots & \ddots & \vdots & \vdots \\ S_1(\alpha_i) & \dots & S_n(\alpha_i) & t^{n-1} \\ S_0(\alpha_i) & \dots & S_{n-1}(\alpha_i) & t^n \end{vmatrix} , \tag{169}$$

form a system of solutions λ_j for the linear system (161), and therefore satisfy (160). □

Due to the significance of the polynomials $Q_i(t)$, we shall give different expressions of them. The first expression, already encountered, is formula (169). Expanding this quasi-determinant by its last column, we compute the coefficients

$$L_{n-k}^{(i)}(A) = (-1)^{n-k-1} \begin{vmatrix} S_n(\alpha_i) & \dots & \boxed{S_{2n-1}(\alpha_i)} \\ \vdots & & \vdots \\ S_{k+1}(\alpha_i) & \dots & S_{k+n}(\alpha_i) \\ S_{k-1}(\alpha_i) & \dots & S_{k+n-2}(\alpha_i) \\ \vdots & & \vdots \\ S_0(\alpha_i) & \dots & S_{n-1}(\alpha_i) \end{vmatrix} \begin{vmatrix} S_{n-1}(\alpha_i) & \dots & S_{2n-2}(\alpha_i) \\ \vdots & & \vdots \\ S_k(\alpha_i) & \dots & \boxed{S_{k+n-1}(\alpha_i)} \\ \vdots & & \vdots \\ S_0(\alpha_i) & \dots & S_{n-1}(\alpha_i) \end{vmatrix} , \ i \ge 1 \ .$$

Recalling that $\sigma(t, \alpha_i) = \lambda(-t, \alpha_i)^{-1}$, we obtain by means of (144) the following expressions in terms of the elementary symmetric functions $\Lambda_j(\alpha_i)$

$$Q_i(t) = \begin{vmatrix} \boxed{\Lambda_0(\alpha_i) - t\,\Lambda_1(\alpha_i) + \dots + (-t)^n \Lambda_n(\alpha_i)} & \Lambda_{n+1}(\alpha_i) & \dots & \Lambda_{2n-1}(\alpha_i) \\ -t\,\Lambda_0(\alpha_i) + \dots + (-t)^n \Lambda_{n-1}(\alpha_i) & \Lambda_n(\alpha_i) & \dots & \Lambda_{2n-2}(\alpha_i) \\ \vdots & \vdots & \ddots & \vdots \\ (-t)^{n-1} \Lambda_0(\alpha_i) + (-t)^n \Lambda_1(\alpha_i) & \Lambda_2(\alpha_i) & \dots & \Lambda_n(\alpha_i) \end{vmatrix} , \tag{170}$$

$$L_k^{(i)}(A) = \begin{vmatrix} \boxed{\Lambda_k(\alpha_i)} & \Lambda_{n+1}(\alpha_i) & \dots & \Lambda_{2n-1}(\alpha_i) \\ \Lambda_{k-1}(\alpha_i) & \Lambda_n(\alpha_i) & \dots & \Lambda_{2n-2}(\alpha_i) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{k-n+1}(\alpha_i) & \Lambda_2(\alpha_i) & \dots & \Lambda_n(\alpha_i) \end{vmatrix} . \tag{171}$$

Also, as shown in the Proof of 8.16, $Q_i(t)$ can be expressed in terms of the entries of A^k, that we denote by $a_{ij}^{(k)}$. Indeed, one has

$$Q_i(t) = \begin{vmatrix} a_{i1}^{(n)} & a_{i2}^{(n)} & \dots & a_{in}^{(n)} & \boxed{1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{i1}^{(1)} & a_{i2}^{(1)} & \dots & a_{in}^{(1)} & t^{n-1} \\ a_{i1}^{(0)} & a_{i2}^{(0)} & \dots & a_{in}^{(0)} & t^n \end{vmatrix} . \tag{172}$$

One can recognize in (172) the exact noncommutative analog of Krylov's expression for the characteristic polynomial of a matrix (*cf.* [Gan]). Thus we find again that, when specialized to the commutative generic matrix, all the polynomials $Q_i(t)$ reduce to the familiar characteristic polynomial (up to the change of variable $u = t^{-1}$). In particular, the leading coefficients of the $Q_i(t)$ provide n noncommutative analogs of the determinant, that we shall call the n *pseudo-determinants* of the generic $n \times n$ matrix A, and denote by $\det_i(A)$, $i = 1, \ldots, n$. They admit the following quasi-determinantal expressions, obtained from (171) and (172)

$$\det_i(A) = \begin{vmatrix} \boxed{\Lambda_n(\alpha_i)} & \Lambda_{n+1}(\alpha_i) & \ldots & \Lambda_{2n-1}(\alpha_i) \\ \Lambda_{n-1}(\alpha_i) & \Lambda_n(\alpha_i) & \ldots & \Lambda_{2n-2}(\alpha_i) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_1(\alpha_i) & \Lambda_2(\alpha_i) & \ldots & \Lambda_n(\alpha_i) \end{vmatrix}, \tag{173}$$

$$\det_i(A) = (-1)^{n-1} \begin{vmatrix} a_{i1}^{(n)} & \ldots & \boxed{a_{ii}^{(n)}} & \ldots & a_{in}^{(n)} \\ \vdots & & \vdots & & \vdots \\ a_{i1}^{(2)} & \ldots & a_{ii}^{(2)} & \ldots & a_{in}^{(2)} \\ a_{i1}^{(1)} & \ldots & a_{ii}^{(1)} & \ldots & a_{in}^{(1)} \end{vmatrix}. \tag{174}$$

We shall illustrate the results of this section on two important examples, namely, the case of the Lie algebra gl_n, and the case of the quantum group $GL_q(n)$.

Example 8.18 We take A to be the matrix E_n of standard generators of $U(gl_n)$ (*cf.* Section 7.5). In this specialization, the pseudo-determinants $\det_i(E_n)$ are all equal to the Capelli determinant. More precisely, we have, using the notations of 7.5

$$Q_i(t) = \det(I - t(E_n + (n-1)I)) = \begin{vmatrix} 1-t(e_{11}+n-1) & -te_{12} & \ldots & -te_{1n} \\ -te_{21} & 1-t(e_{22}+n-2) & \ldots & -te_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -te_{n1} & -te_{n2} & \ldots & 1-te_{nn} \end{vmatrix},$$

for all $i = 1, \ldots, n$. Let $\overline{Q}(u) = u^n\, Q_i(u^{-1}) = \det(uI - (E_n + (n-1)I))$ denote the characteristic polynomial of E_n. By Theorem 8.17, $\overline{Q}(E_n) = 0$. Consider an irreducible finite-dimensional gl_n-module V with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. It follows from Section 7.5 that the coefficients of $\overline{Q}(u)$ belong to the center of $U(gl_n)$, and act upon V as the elementary symmetric fonctions of the commutative variables $\lambda_1 + n - 1$, $\lambda_2 + n - 2, \ldots, \lambda_n$. Therefore, if we denote by $E_n^\lambda = (e_{ij}^{(\lambda)})$ the matrix whose (i,j)-entry is the image of e_{ij} in the module V, we find that

$$\prod_{1\le k\le n} (E_n^\lambda - (\lambda_k + n - k)) = 0\ ,$$

the so-called characteristic identity satisfied by the generators of gl_n.

Example 8.19 Here, A is taken to be the matrix $A_q = (a_{ij})$ of the generators of the quantum group $GL_q(n)$. Recall that the a_{ij} are subject to the following relations

$$a_{ik}a_{il} = q^{-1}a_{il}a_{ik} \quad \text{for } k < l, \quad a_{ik}a_{jk} = q^{-1}a_{jk}a_{ik} \quad \text{for } i < j,$$
$$a_{il}a_{jk} = a_{jk}a_{il} \quad \text{for } i < j, \ k < l,$$
$$a_{ik}a_{jl} - a_{jl}a_{ik} = (q^{-1} - q)\, a_{il}a_{jk} \quad \text{for } i < j, \ k < l \, .$$

In this specialization, the pseudo-determinants $\det_i(A_q)$ are all equal, up to a power of q, to the quantum determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}_q := \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)}\, a_{1\, j_{\sigma(1)}} \dots a_{n\, j_{\sigma(n)}} \, .$$

The other coefficients of the $Q_i(t)$ also specialize to polynomials in the a_{ij}, and one recovers the quantum Cayley-Hamilton theorem proved by Zhang in [Zh]. For instance, when $n = 3$, one obtains

$$Q_1(t) = 1 - (a_{11} + q^{-1}a_{22} + q^{-1}a_{33})\, t$$

$$+ \left(q^{-1} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_q + q^{-1} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}_q + q^{-2} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_q \right) t^2 - q^{-2} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_q t^3 \, ,$$

and for $n = 2$, the Cayley-Hamilton theorem assumes the form

$$A_q^2 - (q^{1/2}a_{11} + q^{-1/2}a_{22}) \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} A_q + (a_{11}a_{22} - q^{-1}a_{12}a_{21}) \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} = 0 \, .$$

In view of these examples, some questions arise naturally. The coefficients $L_j^{(i)}(A)$ involved in the Cayley-Hamilton theorem for the generic noncommutative matrix A are fairly complicated expressions in terms of the entries a_{ij} of A, involving inverses a_{ij}^{-1}, and thus belonging to the skew field generated by the a_{ij}. Moreover, the $L_j^{(i)}(A)$ depend on the index i, that is, the coefficients of the noncommutative characteristic polynomial are no longer scalars but diagonal matrices. It seems to be an interesting problem to investigate the following classes of matrices.

1. The matrices A for which the coefficients $L_j^{(i)}(A)$ are polynomials in the entries a_{ij}.

2. The matrices A for which the coefficients $L_j^{(i)}(A)$ do not depend on i.

As shown by the previous examples, the matrix E_n of generators of $U(gl_n)$ belongs to both classes, while the matrix A_q of generators of $GL_q(n)$ belongs to the first one.

9 Appendix : automata and matrices over noncommutative rings

Identities between quasi-determinants can often be interpreted in terms of *automata*. We present here the basic notions of this theory. More information can be found for instance in the classical references [BR] or [Eil].

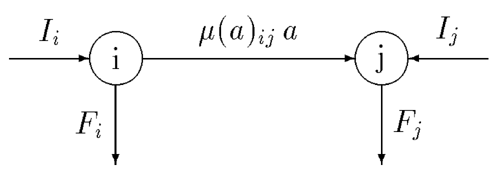
Let K be a field and let A be a noncommutative alphabet. A K-*automaton* of order n over A is a triple $\mathcal{A} = (I, \mu, F)$ where

1. I is a row vector of K^n, called the *initial vector*,
2. μ is a monoid morphism from the free monoid A^* into $M_n(K)$,
3. F a column vector of K^n, called the *final vector*.

One can represent graphically an automaton by a labeled graph (also denoted $\mathcal{A}$) which is defined by

- the set of vertices of $\mathcal{A}$ is $[1, n]$ (they are called the states of $\mathcal{A}$),
- for every $i, j \in [1, n]$ and for every $a \in A$, there is an arrow going from i to j and labeled by $\mu(a)_{ij}\, a$,
- for every vertex $i \in [1, n]$, there is an arrow pointing on i and labeled by I_i,
- for every vertex $i \in [1, n]$, there is an arrow issued from i and labeled by F_i,

with the convention that one does not represent an arrow labelled by 0. In other words, the generic form of the graph $\mathcal{A}$ is the following



Note that the graph can be identified with the automaton since it encodes both the vectors I, F and the matrices $(\mu(a))_{a \in A}$ which completely define the morphism μ. The *behaviour* of $\mathcal{A}$ is the formal noncommutative power series over A defined by

$$\underline{\mathcal{A}} = \sum_{w \in A^*} (\, I\, \mu(w)\, F\,)\, w \quad \in K \ll A \gg \ .$$

The behaviour of $\mathcal{A}$ has a simple graphical interpretation. For every path $\pi = (i_1, \ldots, i_{n+1})$ in the graph $\mathcal{A}$ going from i_1 to i_{n+1} and indexed by the word $w = a_1 \ldots a_n$, we define the cost $c(\pi)$ by

$$c(\pi) = I_{i_1}\, \mu(a_1)_{i_1, i_2} \ldots \mu(a_n)_{i_n, i_{n+1}}\, F_{i_{n+1}} \ .$$

In other words, the cost of a path is just the product of the elements of k that index the different arrows it encounters on its way. Defining the *cost* $c_{\mathcal{A}}(w)$ of a word w relatively to $\mathcal{A}$, as the sum of the costs of all paths indexed by w in $\mathcal{A}$, we have by definition,

$$\underline{\mathcal{A}} = \sum_{w \in A^*} c_{\mathcal{A}}(w)\, w \ .$$

It is not difficult to show that

$$\underline{\mathcal{A}} = I \cdot \left(\sum_{a \in A} \mu(a)\, a \right)^* \cdot F \ ,$$

where the *star* M^* of a matrix M is defined by

$$M^* = (\, I - M\,)^{-1} = \sum_{n \geq 0} M^n \ . \tag{175}$$

It follows that

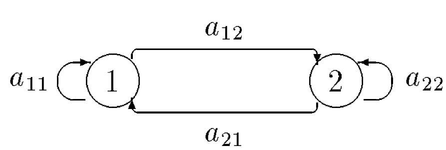
$$I \cdot \left(\sum_{a \in A} \mu(a)\, a \right)^* \cdot F = \sum_{w \in A^*} c_{\mathcal{A}}(w)\, w \ .$$

A series $f \in K << A >>$ is said to be *recognizable* if there exists a K-automaton $\mathcal{A}$ over A for which $f = \underline{\mathcal{A}}$. On the other hand, one can also define a notion of rational noncommutative series in $K << A >>$. A series $f \in K << A >>$ is *rational* in the sense of automata theory if it belongs to the subalgebra $K_{rat} << A >>$, which is the smallest subring of $K << A >>$ which contains $K < A >$ and which is closed under inversion of formal series. One then has the following fundamental theorem of Schützenberger (1961) (see [BR] or [Eil] for a proof).

Theorem 9.1 *A series $f \in K << A >>$ is rational if and only if it is recognizable.*

The link between this notion of rationality and the one considered in Section 8 of the present paper is provided by a result of Fliess ([Fl], see also [Ca]): when the free field is realized as the subfield of the field of Malcev-Neumann series over the free group $F(A)$ generated by the K-alpgebra of $F(A)$, one has $K_{rat} << A >> = K << A >> \cap K \not< A \not>$.

For our purposes, we need only the following particular case. We take $A = \{\, a_{ij},\ 1 \leq i, j \leq n \,\}$ and $\mu(a_{ij}) = E_{ij}$, where E_{ij} denotes the $n \times n$ matrix with only one nonzero entry equal to 1 in position i, j. The graph $\mathcal{A}$ is then the complete oriented graph on n vertices, the arrow from i to j being labelled by a_{ij}. Thus, for $n = 2$, the graph $\mathcal{A}$ is for instance



Denoting also by A the $n \times n$ matrix (a_{ij}), we then have

$$\underline{\mathcal{A}} = I\, A^*\, F \ .$$

In particular, taking all the entries of I and F equal to zero with the exception of one equal to 1, one obtains a graphical interpretation for any entry of the star of a matrix. That is, denoting by $\mathcal{P}_{ij}$ the set of words associated with a path from i to j, one has

$$(A^*)_{ij} = \sum_{w \in \mathcal{P}_{ij}} w \ .$$

This leads to the classical formula (see *e.g.* [BR])

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} (a_{11} + a_{12}\, a_{22}^*\, a_{21})^* & a_{11}^*\, a_{12}\, (a_{22} + a_{21}\, a_{11}^*\, a_{12})^* \\ a_{22}^*\, a_{21}\, (a_{11} + a_{12}\, a_{22}^*\, a_{21})^* & (a_{22} + a_{21}\, a_{11}^*\, a_{12})^* \end{pmatrix} \ . \tag{176}$$

Indeed, the entry $(1,1)$ of the matrix in the right-hand side of (176) represents the set of words labelling a path from 1 to 1 in the automaton $\mathcal{A}$ associated with the generic matrix of order 2. Here, the star of a series s in the a_{ij} with zero constant coefficient is defined, as in (175), by setting $s^* = \sum_{n \geq 0} s^n$.

Observe that formulas (2.2) are exactly the images of (176) under the involutive field automorphism ι of $K \not< A \not>$ defined by

$$\iota(a_{ij}) = \begin{cases} 1 - a_{ii} & \text{if } i = j \\ -a_{ij} & \text{if } i \neq j \end{cases} \tag{177}$$

(see [Co] p. 89), which maps the generic matrix A on $I - A$, so that $\iota(A^*) = A^{-1}$.

As an illustration, let us sketch an automata theoretic proof of Proposition 2.6. The quasi-determinant in the left-hand side of (19) can be written as

$$D = a_{1n} - (a_{11}\ a_{12}\ \ldots\ a_{1,n-1})\ (I_{n-1} - M)^{-1} \begin{pmatrix} a_{2n} \\ a_{3n} \\ \vdots \\ a_{nn} \end{pmatrix}, \tag{178}$$

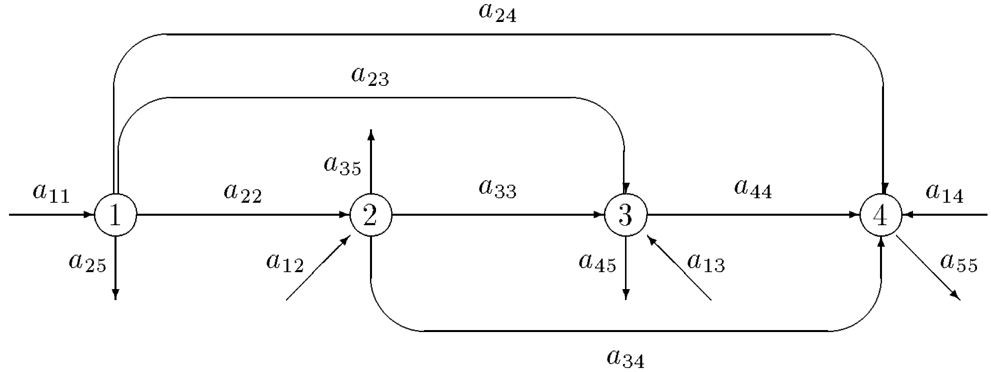
where M denotes the strictly upper triangular matrix defined by

$$M = \begin{pmatrix} 0 & a_{22} & \ldots & a_{2,n-1} \\ 0 & 0 & \ldots & a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Relation (178) shows that D is essentially equal to the behaviour of the automaton $\mathcal{A}_{n-1}$ defined by

- the set of states of $\mathcal{A}_{n-1}$ is $1, 2, \ldots, n-1$,
- the only edges in $\mathcal{A}_{n-1}$ go from i to j with $i < j$ and are labeled by $a_{i+1,j}$,
- each state i is equiped with an initial and a final arrow, respectively labeled by a_{1i} and by $a_{i+1,n}$.

We give below the automaton $\mathcal{A}_4$ which illustrates the general structure of $\mathcal{A}_n$.



It is now clear that

$$D = a_{1n} + \sum_{i<j} a_{1i} \left(\sum \text{ words labelling paths from } i \text{ to } j \text{ in } \mathcal{A}_{n-1} \right) a_{j+1,n} \ ,$$

which is the right-hand side of (19). $\square$

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